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ABSTRACT

The two-dimensional complex sphere $S_1^2 + S_2^2 + S_3^2 = S^2$ forms a homogeneous space under the SL(2,C) group. The little group of a point in this space is the SO(2,C) group or the horosheric group T(2) according to whether $S\neq 0$ or $S=0$. Deformation of the SO (2,C) group into T (2) is investigated and is demonstrated on unitary representations. This deformation is a counterpart to that of the little groups SO(3), E(2), SO(2,1) belonging to the hyperboloid family.

РЕЗЮМЕ

Двумерная комплексная сфера $S_1^2+S_2^2+S_3^2=S^2$ образует однородное пространство относительно группы Лоренца. Малыми группами некоторой точки, находящейся в этом пространстве являются SO(2,C) или группа орисферических трансляций T(2) в зависимости от того, что $S\neq 0$ или $S=0$. В настоящей работе рассмотрена деформация группы SO(2,C) в группу T(2), а также продемонстрирована на соответствующих унитарных представлениях. Вышеуказанная деформация представляет собой аналогию деформаций друг в друга малых групп SO(3), E(2), SO(2,1), принадлежащих к семейству гиперболоида.

KIVONAT

Az $S_1^2 + S_2^2 + S_3^2 = S^2$ egyenlet által leirt kétdimenziós komplex gömb homogén teret képez az SL (2,C) csoporttal szemben. Egy ebben a térben helyetfoglaló pont kiscsoportja az SO (2,C) illetve a T (2) horoszférikus alcsoport, attól függően, hogy $S\neq 0$ vagy $S=0$. Jelen dolgozatban az SO (2,C) csoportnak a T (2) csoportba való deformációját vizsgáljuk és az unitér ábrázolásokon demonstráljuk. Ez a deformáció a hiperboloid családhoz tartozó SO (3), E (2), SO (2,1) kiscsoportok egymásba való deformálásának analogonja.

horospheric group isomorphic to the two-dimensional translation group T(2). To this end a family of homogeneous spaces should be used the little groups of which are apt for demonstration of the deformation process. Since the proper Lorentz group is isomorphic to the connected part of the three-dimensional complex rotation group [5], the two-dimensional complex sphere

$$S_1^2 + S_2^2 + S_3^2 = S^2 \qquad /1.1/$$

/hereafter Σ_S / forms a homogeneous space under the proper Lorentz group, as well as under SL(2,C). The three-dimensional complex vector $\underset{\sim}{S} = (S_1, S_2, S_3)$, which is the self-dual part of a Lorentz covariant antisymmetric tensor $S_{\mu\nu}$ /$\mu,\nu = 0,1,2,3$/ under $A \in SL(2,C)$ transforms as follows

$$\hat{S}' = A\,\hat{S}\,A^{-1} \qquad /1.2/$$

where $\hat{S} = \underset{\sim}{\sigma}\underset{\sim}{S} = \sigma_1 S_1 + \sigma_2 S_2 + \sigma_3 S_3$ and σ_i-s stand for the Pauli matrices.*/

We choose a standard vector on Σ_o as follows

$$\underset{\sim}{S}_o = (0, 0, S) \ . \qquad /1.3/$$

Here S is supposed to be non-zero. The little group of this vector, that is, the subgroup of SL(2,C) satisfying the condition

$$\hat{S}_o = H_o\,\hat{S}_o\,H_o^{-1} \qquad /1.4/$$

is clearly of the form

$$H_o(\varphi) = e^{-i\frac{\varphi}{2}\sigma_3} = \begin{pmatrix} e^{-i\frac{\varphi}{2}} & 0 \\ 0 & e^{i\frac{\varphi}{2}} \end{pmatrix} \qquad /1.5/$$

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*/ Under proper Lorentz transformations, a complex vector $\underset{\sim}{S}$ transforms like $\underset{\sim}{B}+i\underset{\sim}{E}$, where $\underset{\sim}{B}$ and $\underset{\sim}{E}$ are the magnetic and electric field strengths. The invariance of the square $(\underset{\sim}{B}+i\underset{\sim}{E})^2 = B^2-E^2 + 2i\underset{\sim}{B}\underset{\sim}{E}$ is well known from electrodynamics as well.

where $\varphi = \varphi_1 + i\varphi_2$ is a complex angle with a real part describing a rotation about the z-axis and varying in the range $-2\pi \leq \varphi_1 < 2\pi$ and with an imaginary part describing a boost along the z-axis and varying in the range $-\infty < \varphi_2 < \infty$. It follows that this group is $SO(2,C) = SO(2) \times SO(1,1)$.

In a similar way, by choosing the standard vector

$$\underset{\sim}{S}_\infty = (S,\ iS,\ 0)\ , \qquad /S \neq 0/ \qquad /1.6/$$

on the complex sphere of zero radius Σ_0 , we arrive at the horospheric little group isomorphic to the two-dimensional real Euclidean translation group $T(2)$,

$$H_\infty(\varphi) = e^{-i\frac{\varphi}{2}\sigma_+} = \begin{pmatrix} 1 & -i\varphi \\ 0 & 1 \end{pmatrix} \qquad /1.7/$$

where $\sigma_+ = \sigma_1 + i\sigma_2$ and $\varphi = \varphi_1 + i\varphi_2$. In the present case both φ_1 and φ_2 vary from $-\infty$ to ∞ . It is easy to see the validity of the inverse statement, that is, if the little group of a three-dimensional complex vector is $T(2)$ /$SO(2,C)$/ then it is situated on the complex sphere of zero /non--zero/ radius. Here and throughout this paper the $\underset{\sim}{S} = 0$ point is supposed to be excluded from Σ_0 since this point itself is invariant under $SL(2,C)$. Hence by including $\underset{\sim}{S} = 0$ the homogeneity of Σ_0 would be spoiled.

Consider now the vector

$$\underset{\sim}{S}_\tau = \left(\frac{\tau}{1+\tau} S,\ \frac{i\tau}{1+\tau} S,\ \frac{1}{1+\tau} S\right) \qquad /1.8/$$

interpolating between $\underset{\sim}{S}_0$ and $\underset{\sim}{S}_\infty$. Here τ is a real parameter describing the deformation varying in the range

$$0 \lesssim \tau < \infty .$$

The limits of the vector /1.8/ as $\tau \to 0$ and $\tau \to \infty$ are $\underset{\sim}{S}_0$ and $\underset{\sim}{S}_\infty$ as given by Eqs. /1.3/ and /1.6/. Since the length of the vector $\underset{\sim}{S}_\tau$

$$\left[(S_\tau)_1^2 + (S_\tau)_2^2 + (S_\tau)_3^2\right]^{1/2} = \frac{S}{1+\tau} \qquad /1.9/$$

is non-zero for $\tau < \infty$ the little group of $\underset{\sim}{S}_\tau$ is an $SO(2,C)$ group isomorphic to H_0. For $\tau \to \infty$ the little group H_∞ as given by Eq. /1.7/ is obtained. By making use of Eqs. /1.2/ and /1.8/ we get an explicit form of the little group of $\underset{\sim}{S}_\tau$ for an arbitrary value of τ :

$$H_\tau(\varphi) = \exp\left\{-i\frac{\varphi}{2}\left(\sigma_3 + \tau\sigma_+\right)\frac{1}{1+\tau}\right\} = \begin{pmatrix} e^{-i\frac{1}{1+\tau}\frac{\varphi}{2}} & -2i\tau\sin\left(\frac{1}{1+\tau}\frac{\varphi}{2}\right) \\ 0 & e^{i\frac{1}{1+\tau}\frac{\varphi}{2}} \end{pmatrix}. \qquad /1.10/$$

In other words, this subgroup satisfies the equation

$$H_\tau(\varphi)\,\hat{S}_\tau\,H_\tau(\varphi)^{-1} = \hat{S}_\tau \qquad /1.11/$$

with S_τ given by /1.8/. The range of $\varphi = \varphi_1 + i\varphi_2$ in this case is given by the inequalities

$$-2\pi(1+\tau) < \varphi_1 < 2\pi(1+\tau), \qquad -\infty < \varphi_2 < \infty .$$

In the next Section we proceed to an investigation of orbits generated by the above subgroup in the space of complex vectors. In particular, we are interested in the orbits as $\tau\to\infty$.

2. Orbits on the Complex Sphere

According to Eq. /1.9/ the final point of the vector S_τ is situated on the complex sphere $\Sigma_{S/1+\tau}$ of radius $\frac{S}{1+\tau}$ which is non-zero for finite τ but tends to zero as $\tau\to\infty$. At any rate, $\underset{\sim}{S}_\tau$ has the little group as given by Eq. /1.10/. Let us fix the value of τ for the time being and see what a little group $H'_\tau(\varphi)$ is obtained if another standard vector of the same length is chosen instead of $\underset{\sim}{S}_\tau$. The answer is trivial, since as a consequence of the homogeneity of $\Sigma_{S/1+\tau}$ there exists an $A\in SL(2,C)$ which translates S_τ into S'_τ :

$$\hat{S}'_\tau = A\,\hat{S}_\tau\,A^{-1} . \qquad /2.1/$$

It follows then from Eq. /1.11/ that

$$H'_\tau(\varphi)\,\hat{S}'_\tau\,H'_\tau(\varphi)^{-1} = \hat{S}'_\tau \qquad /2.2/$$

where

$$H'_\tau(\varphi) = A\,H_\tau(\varphi)\,A^{-1} \quad . \qquad /2.3/$$

Thus the little group of an arbitrary complex vector $S'_\tau(\varphi)$ is a group conjugate to $H_\tau(\varphi)$ given by Eq./1.10/, that is, $H'_\tau(\varphi)$ is isomorphic to $SO(2,C)$ when $\tau\to\infty$ and isomorphic to $T(2)$ as $\tau\to\infty$.

Now, we are interested in orbits of a complex vector $\underset{\sim}{S}$ under the group /2.3/. It is supposed that the final point of $\underset{\sim}{S}$ is situated on a complex sphere Σ_S of non-zero radius /not to be confused with the sphere $\Sigma_{S/1+\tau}$/. Under the group $H'_\tau(\varphi)$ the vector describes the orbit

$$\hat{S}_\tau(\varphi) = H'_\tau(\varphi)\,\hat{S}\,H'_\tau(\varphi)^{-1} \quad . \qquad /2.4/$$

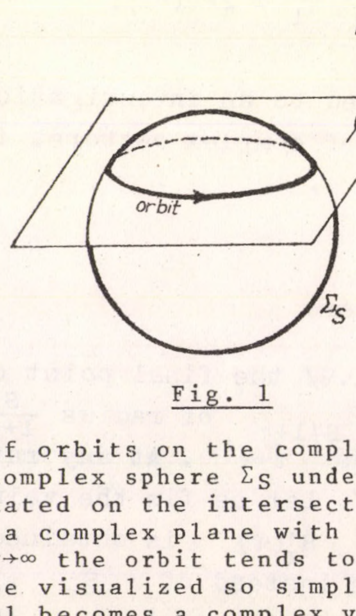

Fig. 1

Illustration of orbits on the complex sphere Σ_S . The orbit on the complex sphere Σ_S under the little group $H'_\tau(\varphi)$ is situated on the intersection of the complex sphere Σ_S and a complex plane with a normal vector $\underset{\sim}{N}_\tau$. In the limit $\tau\to\infty$ the orbit tends to the horosphere, which cannot be visualized so simply since in this case the normal becomes a complex vector of zero length (i.e. $\underset{\sim}{N}_\infty^2 = 0$, though $\underset{\sim}{N}_\infty \neq 0$).

Since $H'(\varphi)\subset SL(2,C)$ the orbit is obviously situated on the sphere Σ_S. Moreover, it will be verified that the orbit lies in a complex plane $\underset{\sim}{S}_\tau(\varphi)\underset{\sim}{N}_\tau = C_\tau$ = const where $\underset{\sim}{N}_\tau$ proves to be indentical with S'_τ given by Eq. /2.1/ Indeed, it follows from Eqs. /2.3/ and /2.4/ that

$$\underset{\sim}{S}_\tau(\varphi)\underset{\sim}{S}'_\tau = \frac{1}{2}\,\mathrm{Tr}\left(\hat{S}_\tau(\varphi)\,\hat{S}'_\tau\right) =$$

$$= \frac{1}{2}\,\mathrm{Tr}\left(H'_\tau(\varphi)\,\hat{S}\,H'_\tau(\varphi)^{-1}\,\hat{S}'_\tau\right) =$$

$$= \frac{1}{2}\,\mathrm{Tr}\left(\hat{S}\,H'_\tau(\varphi)^{-1}\,\hat{S}'_\tau\,H'_\tau(\varphi)\right) = \qquad /2.5/$$

$$= \frac{1}{2}\mathrm{Tr}\left(\hat{S}\,\hat{S}'_\tau\right) = \underset{\sim}{S}\,\underset{\sim}{S}'_\tau \equiv C_\tau = \text{const.}$$

According to this equation one can associate with each orbit generated by the little group $H'_\tau(\varphi)$ a normal vector $\underset{\sim}{S}'_\tau$. The orbit can be given by the homogeneous coordinates $(\underset{\sim}{S}'_\tau, C_\tau)$; nevertheless, apart from the singular case $C_\tau = 0$, one can normalize C_τ to 1 by an appropriate dilatation of S'_τ. As the above statements are independent of the value of τ, we can take the limit $\tau\to\infty$, which produces horospheres. So, according to /2.3/ and /2.4/ the horospheres on $\sum_s$ are orbits described by the horospheric subgroup

$$H'_\infty(\varphi) = \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\begin{pmatrix}1 & -i\varphi\\ 0 & 1\end{pmatrix}\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}^{-1} = A\,H_\infty\,A^{-1} \qquad /\alpha\delta - \gamma\beta = 1/$$

for fixed A . Taking into account Eq. /1.9/ and the fact that transformation /2.1/ leaves the length of $\underset{\sim}{S}_\tau$ unchanged, we get for $\tau\to\infty$

$$(S'_\infty)_1^2 + (S'_\infty)_2^2 + (S'_\infty)_3^2 = 0 \;.$$

Thus, when the $SO(2,C)$ group deforms into the horospheric group as $\tau\to\infty$, the normal vector $\underset{\sim}{N}_\tau \equiv \underset{\sim}{S}'_\tau$ characteristic for the orbits arrives at the complex sphere of zero radius, that is $\underset{\sim}{N}_\infty^2 = \underset{\sim}{S}'^2_\infty = 0$ .It is concluded that horospheres of Σ are determined by the equation $\underset{\sim}{S}\underset{\sim}{N}_\infty = 1$, where $\underset{\sim}{N}_\infty^2 = 0$. */

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*/ The real and imaginary parts of a complex vector on the complex sphere of zero radius are quantities analogous to the field strengths $\underset{\sim}{B}$ and $\underset{\sim}{E}$ of an electromagnetic plane wave, where $\underset{\sim}{B}$ and $\underset{\sim}{E}$ are of the same modulus and are perpendicular to each other: $\underset{\sim}{B}^2 - \underset{\sim}{E}^2 = 0$, $\underset{\sim}{B}\underset{\sim}{E} = 0$.

At this point a remark is in order. Namely, we did not investigate the question whether in Eq. /2.5/ the normal vector of the plane of horospheres is unique up to a factor. From a more detailed investigation which for the sake of brevity is left to the reader, the following can be shown. A single fixed point $\underset{\sim}{S}$ of the space Σ_S is crossed by a one-parametric manifold of horospheres. These are second order curves which, generally speaking, determine unambiguously a plane with a normal of zero length, as indicated above. However, in the manifold of horospheres crossing a fixed point there are two positions where the horosphere degenerates into a complex straight line. These lines can be given in the form

$$\underset{\sim}{S}_\infty(\varphi) = \underset{\sim}{A}\varphi + \underset{\sim}{S} \quad , \quad \underset{\sim}{S}_\infty(\varphi) = \underset{\sim}{B}\varphi + \underset{\sim}{S}$$

with

$$\underset{\sim}{A} = \left(A_1 + iA_2,\, A_1 - iA_2,\, A_3\right) = \left(S\,\frac{S_+^2}{(S+S_3)^2},\ -S,\ S\,\frac{S_+}{S+S_3} \right)$$

$$\underset{\sim}{B} = \left(B_1 + iB_2,\, B_1 - iB_2,\, B_3\right) = \left(-S\,\frac{S_+^2}{(S-S_3)^2},\ S,\ S\,\frac{S_+}{S-S_3} \right) \qquad /S \pm S_3 \neq 0/$$

where the usual $S_\pm = S_1 \pm iS_2$ notation is used. Therefore, each point $\underset{\sim}{S}$ is crossed by two straight horospheres, that are determined by the position of $\underset{\sim}{S}$ alone. These horospheres can be called horospheres of the second kind, as distinguished from those of the first kind, which are in one-to-one correspondence with the vectors of the sphere of zero radius. An analogous situation is encountered in the familiar case of the three-dimensional real one-sheeted hyperboloid [1].

3. Deformation of Unitary Representations

To demonstrate the deformation on unitary representations let us consider the linear fractional mapping of the z-plane which is a factor space $SL(2,C)/\begin{pmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{pmatrix}$:

$$z' = \frac{\alpha z + \beta}{\gamma z + \delta} .$$

In the case of the SO(2,C) subgroup given by /1.5/ this reduces to

$$z' = e^{-i\varphi} z = e^{\varphi_2} e^{-i\varphi_1} z \qquad /3.1/$$

which is a rotation followed by a dilatation. In a similar way, the horospheric transformation on the z-plane takes the form of an Euclidean displacement

$$z' = z - i\varphi = z + (\varphi_2 - i\varphi_1) \quad .$$

The interpolating subgroup given by Eq. /1.10/ accomplishes a transformation similar to that of /3.1/ on the displaced z-plane, i.e.

$$z' + \tau = e^{-i\frac{\varphi}{1+\tau}} (z+\tau) \quad .$$

For realizing unitary representations the representation on the familiar $\varphi(z)$ functions will be used [1]. Action of an element $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ on these functions is defined as

$$T_A \varphi(z) = (-\gamma z + \alpha)^{2j} (-\gamma^* z^* + \alpha^*)^{2k} \varphi\left(\frac{\delta z - \beta}{-\gamma z + \alpha}\right) \qquad /3.2/$$

where

$$j = \frac{1}{2}(j_o - 1 + i\sigma), \quad k = \frac{1}{2}(-j_o - 1 + i\sigma^*). \qquad /3.3/$$

Here j_o takes integer and half-integer values, while σ is an arbitrary complex number. In what follows we restrict ourselves to the principal series of unitary representations for which σ is real. In Eq. /3.2/ representations are defined by displacement from the left which results in the following form of infinitesimal generators:

$$J_+ = J_1 + iJ_2 = -\frac{\partial}{\partial z} \quad , \qquad K_+ = K_1 + iK_2 = 2kz^* - z^{*2}\frac{\partial}{\partial z^*}$$

$$J_- = J_1 - iJ_2 = -2jz + z^2\frac{\partial}{\partial z} \, , \; K_- = K_1 - iK_2 = \frac{\partial}{\partial z^*} \qquad /3.4/$$

$$J_3 = j - z\frac{\partial}{\partial z} \quad , \qquad K_3 = -k + z^*\frac{\partial}{\partial z^*} \quad .$$

These generators are related to the generators of spatial rotations about k^{th}-axis M_k /k = 1,2,3/ and to the generators of boosts along k^{th}-axis N_k as $J_k = \frac{1}{2}(M_k + iN_k)$, $K_k = \frac{1}{2}(M_k - iN_k)$.

Spherical functions in $so(2,C)$ basis satisfy the eigenvalue equations

$$J_3\, \varphi^{o}_{mm^*}(z,z^*) = m\, \varphi^{o}_{mm^*}(z,z^*)$$

$$K_3\, \varphi^{o}_{mm^*}(z,z^*) = m^*\, \varphi^{o}_{mm^*}(z,z^*) \qquad /3.5/$$

where

$$m = \frac{1}{2}(\mu+i\nu)\ , \quad m^* = \frac{1}{2}(\mu-i\nu) \qquad /3.6/$$

with $\mu = 0, \pm\frac{1}{2}, \pm 1, \ldots$ and ν continuous. The above basis is a generalization of finite dimensional spinors to the unitary case where m and m^* correspond to undotted and dotted indices of spinors. Unitary spinors can be succesfully applied to the evaluation of matrix elements of unitary representations of the Lorentz group, namely, they simplify to considerable extent the results obtained in angular momentum basis. [7, 8, 9, 10].

The solution of /3.5/ is

$$\varphi^{o}_{mm^*}(z,z^*) = \frac{1}{2\pi}\, z^{j-m}\, z^{*\,k+m^*} \ . \qquad /3.7/$$

The requirement of single-valuedness on the complex z-plane yields the condition 2μ = integer or, to be strict, μ takes integer and half-integer values along with j_o. The functions /3.7/ are normalized as

$$\langle m'm'^*|mm^*\rangle = \frac{i}{2}\int dz\, dz^*\, \varphi^{o}_{m'm'^*}(zz^*)^{*}\, \varphi^{o}_{mm^*}(z,z^*) = \delta_{\mu'\mu}\, \delta(\nu'-\nu)\ .$$

The horospheric group as given by Eq. /1.7/ is generated by the Hermitean generators M_1-N_2 and M_2+N_1 or - equivalently - by the non--Hermitean generators J_+ and K_-. In this basis spherical functions are solutions of the eigenvalue equations

$$J_+\, \varphi^{\infty}_{mm^*}(z,z^*) = m\, \varphi^{\infty}_{mm^*}(z,z^*)\ ,$$

$$K_-\, \varphi^{\infty}_{mm^*}(z,z^*) = m^*\, \varphi^{\infty}_{mm^*}(z,z^*) \qquad /3.8/$$

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