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ABSTRACT

The parametric resonance in the decay of a system such as a positronium is discussed. It is shown, using Wigner-Weisskopf perturbation theory, that the decay constants depend on the frequency of the external perturbation.

РЕЗЮМЕ

Обсуждается параметрический резонанс в распаде систем, подобных позитронию. При помощи теории возмущения Вигнера и Вайскопфа показано, что постоянные распада существенно зависят от частоты внешнего возмущения.

KIVONAT

Pozitróniumhoz hasonló rendszerek bomlásában fellépő parametrikus rezonanciát tárgyaljuk. Megmutatjuk, hogy a bomlási állandók lényeges módon függenek a külső perturbáció frekvenciájától.

1. INTRODUCTION

Let us consider a system which undergoes a spontaneous radioactive decay other than an electromagnetic transition between the energy levels. An example is the positronium which decays into two or three gamma quanta. What kind of modification in the decay law can be expected when the system undergoes a small classical periodical external perturbation? One is inclined to think that the sole effect consists in the change of the time distribution from a single exponent, characterizing the decay of the ground state, into a sum of exponents

$$p(t) = \sum A_s(\Omega)\, e^{-\gamma_s t}$$

each term of which describes the decay of a single stationary state. The decay constants γ_s are expected to be those for the corresponding levels, while the weights A_s, depending on the frequency Ω , characterize the mixing of the states under the influence of the periodic perturbation. The answer to be given below, however, differs from this intuitive expectation in that the decay constants γ_s in the individual terms of $p(t)$ turn out to depend on the frequency in an essential way. According to the intuitive picture one expects, for example, that at about the resonance frequency $\omega_q - \omega_o$, when the ground state /s=0/ and one of the excited states /s=q/ are involved, $p(t)$ will consist of two different exponents of nearly equal weight. The argument presented here leads to the conclusion that at resonance the distribution $p(t)$ is a <u>single</u> exponent with the decay constant equal to $2^{-1}(\gamma_o+\gamma_q)$. In addition, the following two properties will be indicated: when the external perturbation goes to zero, the effect disappears at a small but finite value, and the statistics of the counts of the decay products differ from the Poisson distribution.

2. THE EXPONENTIALLY DECAYING COMPONENTS

In order to identify the exponentially decaying states of the system the perturbation theory of Wigner and Weisskopf [1] will be applied. Originally this method was conceived to treat the natural broadening of spectral lines observed when the accuracy of the frequency measurement is of the order of γ . According to the uncertainty relation $\Delta\Omega \cdot \Delta t \cong 1$, such an accuracy in the frequency requires the determination of the wave function for times about γ^{-1} , which is beyond the possibilities of the time-dependent perturbation theory of Dirac. Later [2] the method was used in the determination of exponentially decaying combinations of K^o meson states, and this is the aspect we are interested in.

The decay rate in a small interval after the initial time can be calculated using time-dependent perturbation theory. This calculation, however, leads to a unique decay law only if, in the course of the decay, the change of the original state is limited to a decrease of the norm. This condition is met in the case of well separated stationary state, but when linear combinations are involved, the change of the norm is usually accompanied by distortion of the linear combination. Exponentially decaying components are those linear combinations for which such distortions are absent. They can be determined using Wigner-Weisskopf perturbation theory. Since we need the equations of the method in a slightly more general form than they are usually presented, the main steps of the derivation will now be given.

The Hamiltonian of the system is the sum

$$\mathcal{H} = H_o + W(t) + V = H(t) + V$$

where V describes the decay and W is the external perturbation. It will be assumed that the decay products are insensitive to W. The stationary states φ_a and φ_ε of the decaying system and the decay products satisfy the Schrödinger equations

$$H_o \varphi_a = \omega_a \varphi_a \quad ; \quad H_o \varphi_\varepsilon = \varepsilon \varphi_\varepsilon$$

The index ε represents, besides the continuous energy variable, all the quantum numbers which are necessary to specify the states of the decay products.

Let $\psi_i(t) = \overset{+}{U}(t)\, \psi(t)$ be the wave function in the interaction picture. The unitary operator $U(t)$ satisfies the equation

$$\dot{U}(t) = -iH(t)\, U(t) \quad ; \quad U(0) = 1$$

The time-dependent Schrödinger equation can be written as

$$i \frac{\partial \psi_i}{\partial t} = V_i(t)\, \psi_i \qquad /1/$$

where

$$V_i(t) = \overset{+}{U}(t)\ V\ U(t) \qquad /2/$$

The substitution of the expansion

$$\psi_i(t) = \sum_a C'_a(t)\ \varphi_a + \sum_\varepsilon c_\varepsilon(t)\ \varphi_\varepsilon$$

into (1) leads, in the lowest order of V, to

$$i\ \dot{C}'_a(t) = \sum_\varepsilon C_\varepsilon(t)\ \langle a|V_i(t)|\varepsilon\rangle \qquad /3/$$

$$i\ \dot{C}_\varepsilon(t) = \sum_a C'_a(t)\ \langle \varepsilon|V_i(t)|a\rangle \qquad /4/$$

Introducing (2) into (4) and integrating, we have

$$iC_\varepsilon(t) = \sum_{ab} \langle\varepsilon|V|b\rangle \int_0^t e^{i\varepsilon t'}\ \langle b|U(t')|a\rangle\ C'_a(t')\ dt' \qquad /5/$$

Eq.(3) can be presented in the form

$$i\dot{C}'_a(t) = \sum_\varepsilon e^{-i\varepsilon t} \sum_c \langle a|\overset{+}{U}(t)|c\rangle\ \langle c|V|\varepsilon\rangle\ C_\varepsilon(t) \qquad /6/$$

Now the following three steps are to be made:

- introduce (5) into (6),
- make the substitution

$$C'_a(t) = e^{-\left(\frac{1}{2}\gamma + i\mu\right)t}\ C_a(t)$$

with constant γ and μ, and

- change the integration variable t' into $t-t'$.

As a result we arrive at the equation

$$\left(\frac{1}{2}\gamma + i\mu\right) C_a(t) - \dot{C}_a(t) = \sum_{bb_1b_2b_3} \langle a|\overset{+}{U}(t)|b_2\rangle\ \langle b_1|U(t)|b_3\rangle\ \cdot$$

$$\cdot \sum_\varepsilon \langle b_3|D(\varepsilon-\mu)|b\rangle\ \langle b_2|V|\varepsilon\rangle\ \langle\varepsilon|V|b_1\rangle\ C_b(t) \qquad /7/$$

where

$$\langle b_3|D(\varepsilon-\mu)|b\rangle = \int_0^t dt'.e^{-i(\varepsilon-\mu+\frac{i}{2}\gamma)t'}\ \langle b_3|U(-t')|b\rangle$$

For sufficiently large values of t the quantity on the right hand side of this equation is independent of the time t, since the matrix element $\langle b_3|U(-t')|b\rangle$ is a sum of periodic term, and using the relation [3]

$$\int_0^t e^{-i(x+i\gamma)t'}\ dt' = \pi\delta(x) - i\ P\ \frac{1}{x}$$

term by term the t-dependence disappears.

The final step consists in averaging (7) over a time interval T which is large compared to the periods characterizing the motion of the decaying system, but small compared to the lifetime. The average of $\dot{c}_a(t)$ is zero, and we get

$$\left(\frac{1}{2}\gamma + i\mu\right)c_a = \sum_{bb_1b_2b_3} \overline{\langle a|\overset{+}{U}(t)|b_2\rangle \langle b_1|U(t)|b_3\rangle} \cdot \qquad /8/$$

$$\cdot \sum_{\varepsilon} \langle b_3|D(\varepsilon-\mu)|b\rangle \langle b_2|V|\varepsilon\rangle \langle \varepsilon|V|b_1\rangle c_b$$

It is assumed that the average of the product $\langle \overset{+}{U}\rangle \langle U\rangle$ is independent of the time, so that the c_a -s are constants and therefore determine the linear combinations decaying exponentially over time intervals T. When the average is time dependent, no exponentially decaying combinations can be selected.

In the limiting case $W = 0$

$$\langle a|U(t)|b\rangle = e^{-i\omega_a t} \delta_{ab}$$

Therefore

$$\overline{\langle a|\overset{+}{U}(t)|b_2\rangle \langle b_1|U(t)|b_3\rangle} = \delta_{ab_2} \delta_{b_1b_3} \cdot \overline{e^{i(\omega_{b_2}-\omega_{b_1})t}} =$$

$$= \delta_{ab_2} \cdot \delta_{b_1b_3} \cdot \begin{cases} \delta_{b_2b_1} & \text{if} \quad \omega_{b_2} \neq \omega_{b_1} \\ 1 & \text{if} \quad \omega_{b_2} = \omega_{b_1} \end{cases}$$

Substituting this into (8) one finds that for a nondegenerate system the exponentially decaying components are the eigenstates φ_a , while in the degenerate case the exponentially decaying components are defined by a nonhermitian eigenvalue equation.

In the next section eq.(8) will be applied to the case of decay in a periodical external field in the following way: For any given frequency the eigensolutions of (8) will be assumed to decay according to the exponential law, with a decay constant equal to the real part of the corresponding eigenvalue /see eq. (16)/. The weights of these exponents will be given by the projection of the initial state of the system on the corresponding eigensolution /eqs.(17) and (18)/.

3. THE RESONANCE IN THE DECAY LAW

If an external field is present the most interesting case occurs when the frequency is near a resonance. To apply (8) an expression for $\langle a|U(t)|b\rangle$ is needed which can be used at the resonance.

Let $\psi^a(t)$ be the solution of the equation

$$i \frac{\partial\psi}{\partial t} = H(t)\,\psi \qquad /9/$$

satisfying the initial condition

$$\psi^{(a)}(0) = \varphi_a$$

Then

$$\langle b|U(t)|a\rangle = \left(\varphi_b,\ \psi^{(a)}(t)\right)$$

i.e. particular solutions of (9) are necessary to construct U.

In the following we confine ourselves to the case of a two-level system and to an external perturbation of the form

$$\langle a|W(t)|b\rangle = \left(1 - \delta_{ab}\right) W_{ab}\left[e^{i(\Omega t+\varphi)} + e^{-i(\Omega t+\varphi)}\right]$$

Eq. (9) can be written as

$$i\dot{c}_o(t) = \left\{e^{i(\omega_o-\omega_1+\Omega)t+i\varphi} + e^{i(\omega_o-\omega_1-\Omega)t-i\varphi}\right\} W_{o1}\, c_1(t)$$

$$i\dot{c}_1(t) = \left\{e^{i(\omega_1-\omega_o+\Omega)t+i\varphi} + e^{i(\omega_1-\omega_o-\Omega)t-i\varphi}\right\} W_{1o}\, c_o(t) \qquad /10/$$

The $c_a(t)$ -s are defined by

$$\psi(t) = \sum_a c_a(t)\, e^{-i\omega_a t}\, \varphi_a$$

Using the substitution

$$c_a(t) = e^{-i(a\omega+\nu)t}\, C_a$$

where C_a, ν are constants and $\omega = \Omega - (\omega_1 - \omega_o)$ is the deviation from the resonance frequency, (10) can be brought into the from

$$\left.\begin{aligned} e^{i\varphi} W_{o1} C_1 &= \nu C_o \\ e^{-i\varphi} W_{1o} C_o - \omega C_1 &= \nu C_1 \end{aligned}\right\} \qquad /11/$$

Quickly oscillating terms have been omitted, because at $\omega \cong 0$ they lead to small corrections only of the order of $|W_{01}| \cdot (\omega_1 - \omega_0)^{-1}$.

The eigenvalues of (11) are

$$\nu_\sigma = \frac{1}{2}\left[-\omega + \sigma . \sqrt{\omega^2 + 4|W_{01}|^2}\right]$$

with $\sigma = \pm 1$. The corresponding eigenfuctions are given by the expressions

$$C_a^{(+)} = \begin{pmatrix} \cos\frac{\beta}{2} \\ e^{-i(\varphi+\psi)} \sin\frac{\beta}{2} \end{pmatrix} ; \qquad C_a^{(-)} = \begin{pmatrix} \sin\frac{\beta}{2} \\ -e^{-i(\varphi+\psi)} \cos\frac{\beta}{2} \end{pmatrix} \qquad /12/$$

where

$$e^{i\psi} = \frac{W_{01}}{|W_{01}|}$$

$$tg\,\frac{\beta}{2} = \frac{\nu_+}{|W_{01}|}$$

The particular solution $\psi^a(t)$ can be written as

$$\psi^{(a)}(t) = \sum_b C_b^{(a)}(t)\, e^{-i\omega_b t}\, \varphi_b$$

with

$$C_b^{(a)}(t) = \sum_\sigma \overset{*}{C}{}_a^{(\sigma)}\, C_b^{(\sigma)}(t) = \sum_\sigma e^{-i(b\omega+\nu\sigma)t}\, \overset{*}{C}{}_a^{(\sigma)}\, C_b^{(\sigma)}$$

Therefore

$$\langle b|U(t)|a\rangle = \sum_\sigma e^{-i(\omega_b+b\omega+\nu_\sigma)t}\, \overset{*}{C}{}_a^{(\sigma)} \cdot C_b^{(\sigma)} + \sigma\left(\frac{|W_{01}|}{\omega_1 - \omega_0}\right)$$

Using this equation in the lowest order we have

$$\langle b_3|D(\varepsilon-\mu)|b\rangle \cong \delta_{bb_3}\left[\pi\delta(\varepsilon-\bar{\omega}) - i\, P\, \frac{1}{\varepsilon - \bar{\omega}}\right] \qquad /13/$$

It has been assumed that the matrix elements of V and the statistical weights of the final states vary slowly enough for the frequencies ω_0, ω_1 to be replaced by an average $\bar{\omega}$.

Turning to the time average in (8), we can write

$$\overline{\langle a|\overset{+}{U}(t)|b_2\rangle \langle b_1|U(t)|b_3\rangle} =$$

$$= \sum_{\sigma'\sigma} \overset{*}{C}{}^{(\sigma')}_{b_2} C^{(\sigma')}_{a} \overset{*}{C}{}^{(\sigma)}_{b_3} C^{(\sigma)}_{b_1} \cdot \overline{\exp i\left[\omega_{b_2}-\omega_{b_1}+\left(b_2-b_1\right)\omega+\nu_{\sigma'}-\nu_{\sigma}\right]t}$$

Near the resonance the average in this expression is equal to

$$\delta_{b_1 b_2} \overline{\exp i\left(\nu_{\sigma'} - \nu_{\sigma}\right)t} \qquad /14/$$

When $|W_{01}| >> T^{-1}$, the time average here is equal to $\delta_{\sigma'\sigma}$, but in the opposite case, and in the interval $|\omega| << T^{-1}$, it is equal to unity. Between these two values the average is time dependent, and no exponentially decaying components can be indentified.

Let us discuss first the more important case of a "strong" W, wher.

$$\overline{\langle a|\overset{+}{U}(t)|b_2\rangle \langle b_1|U(t)|b_3\rangle} = \delta_{b_1 b_2} \cdot \sum_{\sigma} \overset{*}{C}{}^{(\sigma)}_{b_1} C^{(\sigma)}_{a} \overset{*}{C}{}^{(\sigma)}_{b_3} C^{(\sigma)}_{b_1} \qquad /15/$$

Substituting (15) and (13) into (8), we have

$$\left(\frac{1}{2}\gamma + i\mu\right)C_a = \sum_{\sigma} C^{(\sigma)}_{a} \cdot \sum_{\varepsilon}\left[\pi\delta(\varepsilon-\bar{\omega}) - i\,P\frac{1}{\varepsilon-\bar{\omega}}\right]$$

$$\sum_{b_1} \overset{*}{C}{}^{(\sigma)}_{b_1} C^{(\sigma)}_{b_1} \cdot |\langle b_1|\dot{V}|\varepsilon\rangle|^2 \cdot \sum_{b} \overset{*}{C}{}^{(\sigma)}_{b} C_b$$

This equation is satisfied by putting

$$C_a = C^{(\sigma)}_{a} \quad ; \quad \gamma^{\sigma} = \sum_{b} \overset{*}{C}{}^{(\sigma)}_{b} C^{(\sigma)}_{b} \gamma_b$$

where γ_b is the decay constant of the state φ_b. Using (12), one finds that

$$\gamma^{\sigma}(x) = \frac{1}{2}\left(\gamma_o + \gamma_1\right) + \frac{\sigma}{2} \cdot \frac{x}{\sqrt{1+x^2}}\left(\gamma_o - \gamma_1\right) \qquad /16/$$

where

$$x = \frac{\omega}{2|W_{o1}|}$$

For a state which at $t = 0$ is described by the wave function $\sum C_a \varphi_a$, the weights A_σ are

$$A(x,\varphi) = \left|\sum_{a} \overset{*}{C}_a C^{(\sigma)}_{a}\right|^2 = |C_o|^2 \frac{1}{2}\left[1 + \sigma\frac{x}{\sqrt{1+x^2}}\right] +$$

$$+ |C_1|^2 \frac{1}{2}\left[1 - \sigma\frac{x}{\sqrt{1+x^2}}\right] + \sigma \cdot \frac{C_o C_1}{\sqrt{1+x^2}} \cos(\psi-\varphi-\alpha) \qquad /17/$$

where

$$e^{i\alpha} = \frac{\overset{*}{C}_o C_1}{|C_o C_1|}$$

Besides x, the coefficients A_σ depend on φ as well. This phase can be considered as randomly distributed, therefore

$$\left.\begin{aligned} p(t) &= \frac{1}{2\pi} \int_0^{2\pi} p_\varphi(t)\, d\varphi \\ p_\varphi(t) &= \sum_\sigma A_\sigma(x,\varphi)\; e^{-\gamma^\sigma(x)t} \end{aligned}\right\} \qquad /18/$$

When, for example, the system at $t = 0$ is in the ground state ($C_o=1$, $C_1=0$), the time distribution is given by

$$p(t) = \frac{1}{2}\left(1 + \frac{x}{\sqrt{1+x^2}}\right) e^{-\gamma^+(x)t} + \frac{1}{2}\left(1 - \frac{x}{\sqrt{1+x^2}}\right) e^{-\gamma^-(x)t}$$

Far from the resonance this is the single exponent $\exp(-\gamma_o t)$, while at the resonance ($x = 0$) it is another single exponent $\exp\left[-2^{-1}(\gamma_o + \gamma_1)\, t\right]$. The width of the resonance on the scale ω is of the order of $|W_{o1}|$.

The probability distribution of the number of counts at a time t in the interval Δt is given by

$$P_n = \frac{1}{2\pi} \int_0^{2\pi} d\varphi \cdot \frac{\left[\mu(\varphi)\right]^n}{n!}\; e^{-\mu(\varphi)}$$

with

$$\mu(\varphi) = -\frac{dp_\varphi}{dt} \cdot \Delta t$$

The dispersion is easily seen to be

$$\sigma^2 = \langle n\rangle + \frac{|C_o C_1|^2}{2(1+x^2)} \left[\gamma^+ e^{-\gamma^+ t} - \gamma^- e^{-\gamma^- t}\right]^2 (\Delta t)^2$$

which is larger than for a Poisson distribution, when both C_o and C_1 are different from zero.

Let us turn finally to the case when $|W_{o1}| \ll T^{-1}$. If in this case the same averageing procedure had to be applied, as above, we would have a very narrow resonance. However, as indicated before, in this case the average (14) is equal to $\delta_{b_1 b_2}$, therefore

$$\langle a|\overset{+}{U}(t)|b_2\rangle \langle b_1|U(t)|b_3\rangle = \delta_{b_1 b_2} \delta_{b_2 a} \delta_{b_3 b_1}$$

Substituting this and (13) into (8), we have

$$\left(\frac{1}{2}\gamma + i\mu\right)c_a = \sum_{\varepsilon}\left[\pi\delta(\varepsilon-\bar{\omega}) - i\, P\,\frac{1}{\varepsilon-\bar{\omega}}\right] |\langle a|V|\varepsilon\rangle|^2 \, c_a$$

which is satisfied by the eigenstates φ_a and decay constans γ_a . This means that at this very small but finite perturbation there is no resonance in the decay.

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