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A COMPLETE SET OF FUNCTIONS IN THE QUANTUM MECHANICAL THREE-BODY PROBLEM

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ABSTRACT

A complete set of basis functions for the quantum mechanical three-body problem is chosen in the form of hyperspherical functions. These functions are characterized by quantum numbers corresponding to the chain $O(6) \supset SU(3) \supset O(3)$. Equations are derived to obtain the basis functions in an explicit form.

РЕЗЮМЕ

Для квантовой задачи трех тел выбирается полная система базисных функций в виде гиперсферических функций. Эти функции характеризуются квантовыми числами, соответствующими цепочке $O(6) \supset SU(3) \supset O(3)$. Получены уравнения для определения базисных функций в явном виде.

KIVONAT

A kvantum-mechanikai háromrészecske-probléma teljes sajátfüggvényrendszerét hipergömbfüggvények alakjában választjuk meg, amelyeket az $O(6) \supset SU(3) \supset O(3)$ láncnak megfelelő kvantumszámok jellemeznek. Megadjuk a sajátfüggvények explicit alakban történő meghatározásához szükséges egyenleteket.

$$\vec{\xi} = -\sqrt{\tfrac{3}{2}}\left(\vec{x}_1 + \vec{x}_2\right)$$
$$\vec{\eta} = \frac{1}{\sqrt{2}}\left(\vec{x}_1 - \vec{x}_2\right) \qquad /1.4/$$

for $m_i = 1$. At the same time $\vec{\xi}$ and $\vec{\eta}$ have to fulfill

$$\vec{\xi}^2 + \vec{\eta}^2 = \vec{x}_1^2 + \vec{x}_2^2 + \vec{x}_3^2 = \rho^2 \qquad /1.5/$$

where ρ is the radius of the five-dimensional sphere. The vectors $\vec{x}_i$; can be considered both in the coordinate space and in the momentum space. In the latter case (1,3) means that we are in the center-of-mass frame, and ρ^2 is a quantity proportional to the energy.

The permutations mix up the components of $\vec{\xi}$ and $\vec{\eta}$, and therefore it is useful to consider a six dimensional vector $X = \begin{pmatrix}\xi\\ \eta\end{pmatrix}$, for which we have

$$P_{12}\begin{pmatrix}\xi\\ \eta\end{pmatrix} = \begin{pmatrix}\xi\\ -\eta\end{pmatrix},$$

$$P_{13}\begin{pmatrix}\xi\\ \eta\end{pmatrix} = \begin{pmatrix}-1/2 & -\sqrt{3}/2\\ -\sqrt{3}/2 & 1/2\end{pmatrix}\begin{pmatrix}\xi\\ \eta\end{pmatrix}, \qquad P_{23}\begin{pmatrix}\xi\\ \eta\end{pmatrix} = \begin{pmatrix}-1/2 & \sqrt{3}/2\\ \sqrt{3}/2 & 1/2\end{pmatrix}\begin{pmatrix}\xi\\ \eta\end{pmatrix} \qquad /1.6/$$

From these formulae it is clear, that the permutations appear as some rotations in the six dimensional space. On the vectors X_i $(i = 1, \dots 6)$ one can build up the group $O(6)$, for which (1,5) can be considered as the invariance condition. The 15 generators of $O(6)$ are

$$a_{ik} = X_i \frac{\partial}{\partial X_k} - X_k \frac{\partial}{\partial X_i} \qquad /1.7/$$

$i, k = 1, \dots, 6.$

Further, we introduce the complex vectors

$$\vec{z} = \vec{\xi} + i\vec{\eta}$$
$$\vec{z}^* = \vec{\xi} - i\vec{\eta} \qquad /1.8/$$

the permutation properties of which are especially simple:

$$P_{12}z = z^* \ , \quad P_{13}z = z^* e^{-2/3\pi i} \ , \quad P_{23}z = z^* e^{2/3\pi i}$$

$$P_{12}z^* = z \ , \quad P_{13}z^* = ze^{2/3\pi i} \ , \quad P_{23}z^* = ze^{-2/3\pi i} \qquad /1.9/$$

For z and z^* the condition (1,5) takes the form $\xi^2 + \eta^2 = |z|^2 = \rho^2$, and can be considered as the invariant of the group SU(3) . In other words, on the vectors z , z^* the group SU(3) can be constructed. The SU(3) generators are, as usual:

$$A_{ik} = iz_i \frac{\partial}{\partial z_k} - iz_k^* \frac{\partial}{\partial z_i^*} \qquad /1.10/$$

$$i,k = 1,2,3$$

The generators of SU(3) and O (6) are connected in the following way:

$$A_{ik} = \frac{i}{2}\left[a_{ik} + i\left(a_{i+3,k} - a_{i,k+3}\right) + a_{i+3,k+3}\right] \qquad /1.11/$$

I.2. Coordinates. Parametrization

In order to complete the parametrization, let's consider a triangle, the vertices of which are determined by three particles. The situation of the plane of this triangle in the space will be characterized by the unit vectors $\vec{\ell}_1$, $\vec{\ell}_2$.

$$\ell_1^2 = \ell_2^2 = 1 \ , \qquad \vec{\ell}_1 \vec{\ell}_2 = 0 \qquad /1.12/$$

They form together with $\vec{\ell} = \vec{\ell}_1 \times \vec{\ell}_2$ the moving coordinate system, the orientation of which to the fixed system of coordinates we describe by the Euler angles φ_1, Θ, φ_2 .

$$\vec{\ell}_1 = \{-\sin\varphi_1 \sin\varphi_2 + \cos\varphi_1 \cos\varphi_2 \cos\Theta; -\sin\varphi_1 \cos\varphi_2 + \cos\varphi_1 \sin\varphi_2 \cos\Theta; -\cos\varphi_1 \sin\Theta\}$$

$$\vec{\ell}_2 = \{-\cos\varphi_1 \sin\varphi_2 - \sin\varphi_1 \cos\varphi_2 \cos\Theta; -\cos\varphi_1 \cos\varphi_2 + \sin\varphi_1 \sin\varphi_2 \cos\Theta; \sin\varphi_1 \sin\Theta\}$$

$$\vec{\ell} = \{-\cos\varphi_2 \sin\Theta; \ \sin\varphi_2 \sin\Theta; \ -\cos\Theta\} \qquad /1.13/$$

Vectors $\vec{\ell}_1$ and $\vec{\ell}_2$ are connected with $\vec{z}$ in the following way:

$$\vec{z} = \frac{\rho}{\sqrt{2}} e^{-i\frac{\lambda}{2}} \left(e^{i\frac{a}{2}} \vec{\ell}_1 + ie^{-i\frac{a}{2}} \vec{\ell}_2\right) \qquad /1.14/$$

where $0 \leqslant a \leqslant \pi$, $0 \leqslant \lambda \leqslant 2\pi$.

The parameters λ and a characterize the form of the triangle /except the similarity transformation, which can be excluded putting ρ = const./ Note, that the parametrization is chosen in such a way, that we can separate the two possible types of motion of the triangle: the spatial rotations and the deformations. That can be easily seen for example, if we rewrite vectors $\vec{\xi}$ and $\vec{\eta}$ in the form

$$\vec{\xi} = \frac{\rho}{\sqrt{2}}\left(\cos\frac{a-\lambda}{2}\vec{\ell}_1 + \sin\frac{a+\lambda}{2}\vec{\ell}_2\right)$$
$$\vec{\eta} = \frac{\rho}{\sqrt{2}}\left(\sin\frac{a-\lambda}{2}\vec{\ell}_1 + \cos\frac{a+\lambda}{2}\vec{\ell}_2\right) \qquad /1.15/$$

However, these expressions can not be obtained as products of functions of the Euler angles and functions of the coordinates related to the deformations /they are, in fact, sums of such functions/. This feature corresponds to the connection between rotations and deformations. To make the picture clearer, consider the case of a non-rotating triangle. We need for that purpose the expressions

$$\xi^2 = \frac{\rho^2}{2}\left(1 + \sin a \sin\lambda\right)$$
$$\eta^2 = \frac{\rho^2}{2}\left(1 - \sin a \sin\lambda\right) \qquad /1.16/$$

and

$$\vec{\xi}\vec{\eta} = \frac{\rho^2}{2}\sin a \cos\lambda \qquad /1.17/$$

The angle Θ between vectors $\vec{\xi}$ and $\vec{\eta}$

$$\vec{\xi}\vec{\eta} = |\xi||\eta| \cos\Theta$$

can be written in terms of the variables λ and a as

$$\cos\Theta = \frac{\cos\lambda \sin a}{\sqrt{1 - \sin^2\lambda \sin^2 a}} \qquad /1.18/$$

Note, that the components of the moment of inertia are

$$\rho^2\sin^2\left(\frac{a}{2} - \frac{\pi}{4}\right), \rho^2\cos^2\left(\frac{a}{2} - \frac{\pi}{4}\right), \rho^2 \qquad /1.19/$$

Thus it is obvious, that, if a = const, the variations of λ lead to such deformations of the triangle, which do not affect the values of momenta of inertia.

Let us return to the parametrization. In some calculations it will be useful to apply z in the form

$$z = \rho e^{-i\frac{\lambda}{2}}\left(\cos\frac{a}{2}\,\vec{\ell}_+ + i\,\sin\frac{a}{2}\,\vec{\ell}_-\right) \qquad /1.20/$$

where

$$\vec{\ell}_+ = \frac{1}{\sqrt{2}}\left(\vec{\ell}_1 + i\vec{\ell}_2\right) \quad , \quad \vec{\ell}_- = \frac{1}{\sqrt{2}}\left(\vec{\ell}_1 - i\vec{\ell}_2\right)$$

$$\vec{\ell}_+\vec{\ell}_- = 1 \ , \ \ell_+^2 = \ell_-^2 = 0 \ , \ \vec{\ell}_o = \left(\vec{\ell}_+ \times \vec{\ell}_-\right) = -i\vec{\ell} \qquad /1.21/$$

The components of $\vec{\ell}_+$ and $\vec{\ell}_-$ can be expressed in terms of the Wigner D-functions, defined as

$$D^{\ell}_{mm}(\varphi_1\Theta\varphi_2) = e^{-i(m\varphi_1+n\varphi_2)}\ P^{\ell}_{mn}(\cos\Theta) \qquad /1.22/$$

in the following way:

$$D^{1}_{mn}(\varphi_1\Theta\varphi_2) = \vec{\ell}_m\,\vec{k}_n = \ell_m^{(n)} \qquad /1.23/$$

Here $\vec{\ell}_m$ and $\vec{k}_n$ are unit vectors corresponding to the moving and the fixed coordinate systems respectively. Using the form (1,20) it is obvious, that the components of $\vec{z}$ and $\vec{z}^*$ can be written as

$$z_M=\rho\left(D^{1/2}_{1/2,-1/2}(\lambda,a,0);\ D^{1}_{-1,M}(\varphi_1\Theta\varphi_2) + D^{1/2}_{1/2,1/2}(\lambda,a,0)\ D^{1}_{1,M}(\varphi_1\Theta\varphi_2)\right) \quad /1.24/$$

$$z^*_M=\rho\left(D^{1/2}_{-1/2,-1/2}(\lambda,a,0)\ D^{1}_{-1,M}(\varphi_1\Theta\varphi_2) - D^{1/2}_{-1/2,1/2}(\lambda,a,0)\ D^{1}_{1,M}(\varphi_1\Theta\varphi_2)\right) \quad /1.25/$$

II. GENERATORS AND CASIMIR OPERATORS IN TERMS OF THE ANGULAR VARIABLES.

2.1. The Choice of Quantum Numbers

The theory of spherical functions, which form the basis in the case of a two-body system, is well known. If one intends to develop an analogous theory of harmonic functions for three particles, it is natural to use angular variables on the five dimensional sphere, and construct the wanted functions in terms of these variables.

Introducing angular variables, we separate the similarity transformations, and consider the group of those transformations only, under which the sum of squared coordinates of the three particles, i.e. the radius of the five dimensional sphere, remains constant.

Consider now a triangle, the vertices of which are determined by three particles. If we exclude similarity transformations, two types of transformations are left: rotations in the ordinary three-dimensional space which are described by the group $O(3)$, and deformations of the triangle.

Now, it is obvious, that different forms of a deforming, non-rotating triangle can be considered as the projections onto its plane of all the possible positions of a rotating rigid triangle. Dealing with both the rotations and deformations, one can say, that all transformations of a triangle besides the similarity transformations can be described by the projections onto the three-dimensional space of a rigid triangle which is rotating in the four-dimensional space. That means, that an arbitrary motion of three particles is equivalent to the rotation of a triangle of unchanging form in the four-dimensional space, and its similarity transformations.

The representations on the five-dimensional sphere of both the group $O(6)$ and its reduction to $SU(3)$ involve the representation of the permutation group S_3. That's why this description is extremely convenient for the system of three equivalent particles.

For the classification of a three-particle state one needs five quantum numbers. Thus it is natural to deal with $SU(3)$ symmetry, in case of which we dispose exactly of the necessary 5 quantum numbers. We have to separate from the $SU(3)$ generator (1,10) the antisymmetric tensor - the generator of the rotation group $O(3)$:

$$J_{ik} = \frac{1}{2}\left(A_{ik} - A_{ki}\right) = \frac{1}{2}\left(iz_i \frac{\partial}{\partial z_k} - iz_k \frac{\partial}{\partial z_i} + iz_i^* \frac{\partial}{\partial z_k^*} - iz_k^* \frac{\partial}{\partial z_i^*}\right) \qquad /2.1/$$

The remaining symmetric part

$$B_{ik} = \frac{1}{2}\left(A_{ik} + A_{ki}\right) = \frac{1}{2}\left(iz_i \frac{\partial}{\partial z_k} + iz_k \frac{\partial}{\partial z_i} - iz^* \frac{\partial}{i\partial z_k^*} - iz_k^* \frac{\partial}{\partial z_i^*}\right) \qquad /2.2/$$

is the generator of the group of deformations of the triangle which turns out to be locally isomorphic with the rotation group. Finally, we introduce a scalar operator

$$N = \frac{1}{2i}\, Sp\, A = \frac{1}{2}\sum_k \left(z_k \frac{\partial}{\partial z_k} - z_k^* \frac{\partial}{\partial z_k^*}\right) \qquad /2.3/$$

To classify the three-body system, we choose the following quantum numbers:

$$K,\ J,\ M,\ \nu,\ \Omega\ .$$

Here $K(K+4)$ is the eigenvalue of the Laplace operator on the five dimensional sphere /quadratic Casimir operator for SU(3) /, $J(J+1)$ - the eigenvalue of the square of the angular momentum operator $J^2 = 4 \sum_{i>k} J_{ik}^2$; M- the eigenvalue of $J_3 = 2J_{12}$ and ν - the eigenvalue of N. Although N is not a Casimir operator of SU(3), the representation might be characterized by means of its eigenvalue, because, as it can be seen, the eigenvalue of the Casimir operator of third order can be written as a combination of K and ν. /If the harmonic function belongs to the SU(3) representation (p,q), then $K = p+q$, $\nu = \frac{1}{2}(p-q)$ /.

The fifth quantum number is not included in any of the considered subgroups; we take it from O(6) and define it as the eigenvalue of the operator

$$\hat{\Omega} = \sum_{i,k,\ell} J_{ik} B_{k\ell} J_{\ell i} = \mathrm{Sp}\, JBJ \qquad /2.5/$$

This cubic generator was first introduced by Racah [16] .

2.2. The Laplace Operator

We have now to write down the operators, the eigenvalues of which we are looking for. First of all let us construct the Laplace operator. We could do that by a straightforward calculation of

$$\Delta = |A_{ik}|^2 \qquad /2.6/$$

but there is a simpler way. We calculate

$$d\vec{z} = \frac{1}{\rho}\vec{z}\, d\rho - \frac{i}{2}\vec{z}\, d\lambda + \frac{1}{2} e^{-i\lambda}(\vec{\ell} \times \vec{z}^*) da - (d\vec{\omega} \times \vec{z}) \qquad /2.7/$$

This rather simple expression is obtained by introducing the infinitesimal rotation $d\vec{\omega}$. Its projections onto the fixed coordinates $\vec{k}_1 = (1, 0, 0)$, $\vec{k}_2 = (0, 1, 0)$, $\vec{k}_3 = (0, 0, 1)$ can be expressed in terms of the Euler angles in a well-known form:

$$\begin{aligned} d\omega_1 &= \cos\varphi_2 \sin\Theta\, d\varphi_1 - \sin\varphi_2\, d\Theta \\ d\omega_2 &= -\sin\varphi_2 \sin\Theta\, d\varphi_1 - \cos\varphi_2\, d\Theta \qquad /2.8/ \\ d\omega_3 &= \cos\Theta\, d\varphi_1 + d\varphi_2 \end{aligned}$$

We shall use the expressions of the infinitesimal rotations about the rotating axes as well; they are defined as

$$d\Omega_i = \vec{\ell}_i\, d\vec{\omega} \qquad /2.9/$$

From (2,7) one easily gets

$$ds^2 = |d\vec{z}|^2 = g_{ik}\, q^i\, q^k =$$

$$= \rho^2 \left[\frac{1}{4}\, da^2 + \frac{1}{4}\, d\lambda^2 + \frac{1}{2}\, d\Omega_1^2 + \frac{1}{2}\, d\Omega_2^2 + d\Omega_3^2 - \right. \qquad /2.10/$$

$$\left. -\sin a\; d\Omega_1\, d\Omega_2 - \cos a\; d\Omega_3\, d\lambda \right] + d\rho^2$$

Since the similarity transformations are of no interest to us, we can from now on put $\rho = 1$.

The expression (2,10) determines the components of the metric tensor g_{ik}, thus it becomes easy to calculate the Laplace operator:

$$\Delta' = \frac{1}{4}\,\Delta = \frac{1}{4}\,\frac{1}{\sqrt{g}}\,\frac{\partial}{\partial q^i}\, g^{ik}\, \sqrt{g}\,\frac{\partial}{\partial q^k} =$$

$$= \frac{\partial^2}{\partial a^2} + 2\mathrm{ctg}\; 2a\frac{\partial}{\partial a} + \frac{1}{\sin^2 a}\left(\frac{\partial^2}{\partial \lambda^2} + \cos a\,\frac{\partial^2}{\partial\lambda\,\partial\Omega_3} + \frac{1}{4}\,\frac{\partial^2}{\partial\Omega_3^2}\right) + \qquad /2.11/$$

$$+ \frac{1}{2\cos^2 a}\left[\frac{\partial^2}{\partial\Omega_1^2} + \sin a\left(\frac{\partial^2}{\partial\Omega_1\,\partial\Omega_2} + \frac{\partial^2}{\partial\Omega_2\,\partial\Omega_1}\right) + \frac{\partial^2}{\partial\Omega_2^2}\right]$$

The explicit form of the operator N is

$$N = i\,\frac{\partial}{\partial\lambda} \qquad /2.12/$$

If a harmonic function ϕ is an eigenfunction of Δ, it has to fulfill

$$\Delta\phi = -K(K+4)\,\phi \qquad /2.13/$$

and

$$N\phi = \nu\phi \qquad /2.14/$$

Rewriting /2,11/ in terms of the Euler angles, we obtain the Laplacian in the form

$$\Delta' = \Delta_a - \mathrm{tg}\; a\,\frac{\partial}{\partial a} + \frac{1}{2\cos^2 a}\left(\Delta_\Theta - \frac{\partial^2}{\partial\varphi_1^2}\right) -$$

$$- \frac{\sin a}{2\cos^2 a}\left[\cos 2\varphi_1\left(\frac{1+\cos^2\Theta}{\sin^2\Theta}\,\frac{\partial}{\partial\varphi_1} - 2\,\frac{\cos\Theta}{\sin^2\Theta}\,\frac{\partial}{\partial\varphi_2} - 2\mathrm{ctg}\Theta\,\frac{\partial^2}{\partial\varphi_1\,\partial\Theta} + \right.\right. \qquad /2.15/$$

$$\left.\left. + 2\,\frac{1}{\sin\Theta}\,\frac{\partial^2}{\partial\varphi_2\,\partial\Theta}\right) + \sin 2\varphi_1\left(\Delta_\Theta - \frac{\partial^2}{\partial\varphi_1^2} - 2\,\frac{\partial^2}{\partial\Theta^2}\right)\right]$$

where Δ_a and Δ_Θ are $O(3)$ Laplace operators:

$$\Delta_a = \frac{\partial^2}{\partial a^2} + \mathrm{ctg}\; a\,\frac{\partial}{\partial a} + \frac{1}{\sin^2 a}\left(\frac{\partial^2}{\partial\lambda^2} + \cos a\,\frac{\partial^2}{\partial\lambda\,\partial\Omega_3} + \frac{1}{4}\,\frac{\partial^2}{\partial\Omega_3}\right) \qquad /2.16/$$

$$\Delta_\Theta = \frac{\partial^2}{\partial\Theta^2} + \mathrm{ctg}\Theta\,\frac{\partial}{\partial\Theta} + \frac{1}{\sin^2\Theta}\left(\frac{\partial^2}{\partial\varphi_1^2} - 2\cos\Theta\,\frac{\partial^2}{\partial\varphi_1\,\partial\varphi_2} + \frac{\partial^2}{\partial\varphi_2^2}\right) \qquad /2.17/$$

The form (2,15) can be obtained from the Laplacian calculated in [9] by a unitary transformation.

2.3. The Generators J_{ik} and B_{ik}

To get the generators directly from $d\vec{z}$, one have to invert a 5 x 5 matrix in the case of three particles. That requires rather a long calculation, which is getting hopeless for a larger number of particles. Instead of performing the straightforward calculation, we obtain the wanted expressions in the following way. Let us first consider J_{ik}, or, to be precise, a component of it, for example J_{12}. We introduce a parameter σ_{ik} which define the displacement along the trajectory which corresponds to the action of the operator J_{ik}. Thus, formally we can write

$$J_{12} = \frac{1}{2}\left(iz_1 \frac{\partial}{\partial z_2} - iz_2 \frac{\partial}{\partial z_1} + iz_1^* \frac{\partial}{\partial z_2^*} - iz_2^* \frac{\partial}{\partial z_1^*}\right) \equiv \frac{\partial}{\partial \sigma_{12}} \qquad /2.18/$$

Acting with J_{12} on the vectors $\vec{z}$ and $\vec{z}$

$$J_{12}\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -iz_2 \\ iz_1 \\ 0 \end{pmatrix}, \qquad J_{12}\begin{pmatrix} z_1^* \\ z_2^* \\ z_3^* \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -iz_2^* \\ iz_1^* \\ 0 \end{pmatrix} \qquad /2.19/$$

we see, that σ_{12} has to be imaginary. In the following, we will make use of the equations

$$\vec{z}\, J_{12}\, \vec{z} = 0\,, \qquad \vec{z}^*\, J_{12}\, \vec{z}^* = 0 \qquad /2.20/$$

$$\vec{z}^* J_{12}\, \vec{z} = \frac{i}{2}\left(\vec{z} \times \vec{z}^*\right)_3 \qquad /2.21/$$

$$\vec{\ell}\, J_{12}\, \vec{z} = -\frac{i}{2}\left(\vec{\ell} \times \vec{z}\right)_3 \qquad /2.22/$$

Using the expression /2,7/ for $d\vec{z}$, we can write

$$J_{12}\vec{z} = \frac{\partial \vec{z}}{\partial \sigma_{12}} = -\frac{i}{2}\,\vec{z}\,\frac{d\lambda}{d\sigma_{12}} + \frac{1}{2}\,e^{-i\lambda}\left(\vec{\ell}\times z^*\right)\frac{da}{d\sigma_{12}} - \left(\frac{d\vec{\omega}}{d\sigma_{12}} \times \vec{z}\right) \qquad /2.23/$$

and

$$J_{12}\vec{z}^* = \frac{\partial \vec{z}^*}{\partial \sigma_{12}} = \frac{i}{2}\,\vec{z}^*\,\frac{d\lambda}{d\sigma_{12}} + \frac{1}{2}\,e^{i\lambda}\left(\vec{\ell}\times\vec{z}\right)\frac{da}{d\sigma_{12}} - \left(\frac{d\vec{\omega}}{d\sigma_{12}} \times \vec{z}^*\right) \qquad /2.24/$$

(here $-\frac{i\vec{z}}{2}$ is $\frac{d\vec{z}}{d\lambda}$ etc.) If we now obtain the derivatives which are included in the right-hand side of the latter equations, then we can express J_{12} in terms of the new variables. Substituting

$$\left(\vec{\ell} \times \vec{z}^*\right) = ie^{i\frac{\lambda}{2}}\left(\cos\frac{a}{2}\,\vec{\ell}_- + i\sin\frac{a}{2}\,\vec{\ell}_+\right) \qquad /2.25/$$

$$\left(\vec{\ell} \times \vec{z}\right) = -ie^{-i\frac{\lambda}{2}}\left(\cos\frac{a}{2}\,\vec{\ell}_+ - i\sin\frac{a}{2}\,\vec{\ell}_-\right) \qquad /2.26/$$

$$z^2 = ie^{-i\lambda}\sin a\ ,\qquad z^{*2} = -ie^{i\lambda}\sin a \qquad /2.27/$$

in /2,23/ and /2,24/ , we get from (2,20)

$$\frac{da}{d\sigma_{12}} = \frac{d\lambda}{d\sigma_{12}} = 0\ . \qquad /2.28/$$

Similarly, /2,21/ gives

$$\frac{d\Omega_3}{d\sigma_{12}} = -\frac{i}{2}\,\ell^{(3)}. \qquad /2.29/$$

and finally, /2,22/ leads to

$$\frac{d\Omega_1}{d\sigma_{12}} = -\frac{i}{2}\,\ell_1^{(3)} \qquad /2.30/$$

$$\frac{d\Omega_2}{d\sigma_{12}} = -\frac{i}{2}\,\ell_2^{(3)} \qquad /2.31/$$

where $\ell_i^{(k)}$ denotes the k-th component of vector $\vec{\ell}_i$.
Thus, we obtain

$$J_{12} = -\frac{i}{2}\left[\ell_1^{(3)}\frac{\partial}{\partial\Omega_1} + \ell_2^{(3)}\frac{\partial}{\partial\Omega_2} + \ell^{(3)}\frac{\partial}{\partial\Omega_3}\right] = -\frac{i}{2}\frac{\partial}{\partial\omega_3} \qquad /2.32/$$

Similarly

$$J_{23} = -\frac{i}{2}\left[\ell_1^{(1)}\frac{\partial}{\partial\Omega_1} + \ell_2^{(1)}\frac{\partial}{\partial\Omega_2} + \ell^{(1)}\frac{\partial}{\partial\Omega_3}\right] = -\frac{i}{2}\frac{\partial}{\partial\omega_1} \qquad /2.33/$$

$$J_{31} = -\frac{i}{2}\left[\ell_1^{(2)}\frac{\partial}{\partial\Omega_1} + \ell_2^{(2)}\frac{\partial}{\partial\Omega_2} + \ell^{(2)}\frac{\partial}{\partial\Omega_3}\right] = -\frac{i}{2}\frac{\partial}{\partial\omega_2} \qquad /2.34/$$

The general expression for the angular momentum operator will have the form

$$\frac{1}{2}J_k \equiv J_{ij} = -\frac{i}{2}\,\varepsilon_{ijk}\left[\ell_1^{(k)}\frac{\partial}{\partial\Omega_1} + \ell_2^{(k)}\frac{\partial}{\partial\Omega_2} + \ell^{(k)}\frac{\partial}{\partial\Omega_3}\right] = -\frac{i}{2}\frac{\partial}{\partial\omega_k} \qquad /2.35/$$

The components of this operator fulfill the commutation relations

$$\left[J_{ik}, J_{jl}\right] = \frac{i}{2}\left(J_{i\ell}\,\delta_{jk} - J_{jk}\,\delta_{i\ell}\right) - \frac{i}{2}\left(J_{ij}\,\delta_{k\ell} - J_{\ell k}\,\delta_{ij}\right) \qquad /2.36/$$

Finally, the square of the angular momentum operator is

$$J^2 = \left(\frac{\partial^2}{\partial\Omega_1^2} + \frac{\partial^2}{\partial\Omega_2^2} + \frac{\partial^2}{\partial\Omega_3^2}\right) = \Delta_\Theta \qquad /2.37/$$

We could, of course, get directly the expressions for J_k. However, we wanted to prove the method, which is necessary to calculate B_{ik}. Let's consider

$$B_{12} = \frac{1}{2}\left(iz_1\frac{\partial}{\partial z_2} + iz_2\frac{\partial}{\partial z_1} - iz_1^*\frac{\partial}{\partial z_2^*} - iz_2^*\frac{\partial}{\partial z_1^*}\right) \equiv \frac{\partial}{\partial\beta_{12}} \qquad /2.38/$$

From the action of B_{12} on $\vec{z}$ and $\vec{z}^*$

$$B_{12}\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} iz_2 \\ iz_1 \\ 0 \end{pmatrix}, \qquad B_{12}\begin{pmatrix} z_1^* \\ z_2^* \\ z_3^* \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -iz_2^* \\ -iz_1^* \\ 0 \end{pmatrix} \qquad /2.39/$$

it is obvious, that β_{12} is real. Making use of

$$\vec{z}\, B_{12}\, \vec{z} = iz_1 z_2 \quad , \quad \vec{z}^*\, B_{12}\, \vec{z}^* = -iz_1^* z_2^* \qquad /2.40/$$

$$\vec{z}^*\, B_{12}\, \vec{z} = \frac{i}{2}\left(z_1^*\, z_2 + z_1\, z_2^*\right) \qquad /2.41/$$

$$\vec{\ell}\, B_{12}\, \vec{z} = \frac{i}{2}\left(\ell^{(1)}\, z_2 + \ell^{(2)}\, z_1\right) \qquad /2.42/$$

and of /2,7/ , /2,25/ and /2,26/ , and following a procedure, similar to that in the case of J_{ik}, we obtain the generator B_{ik} of the group of deformations of the triangle:

$$B_{ik} = \left(b_{11}^{(ik)} - b_{22}^{(ik)}\right)\frac{\partial}{\partial a} - \left(b_{11}^{(ik)} + b_{22}^{(ik)}\right)\frac{\partial}{\partial \lambda} -$$
$$- 2b_{12}^{(ik)}\left(\frac{1}{\sin a}\frac{\partial}{\partial \lambda} + \frac{1}{2}\operatorname{ctg} a \frac{\partial}{\partial \Omega_3}\right) - \qquad /2.43/$$
$$- b_{13}^{(ik)}\left(\operatorname{tg} a \frac{\partial}{\partial \Omega_2} + \frac{1}{\cos a}\frac{\partial}{\partial \Omega_1}\right) - b_{23}^{(ik)}\left(\operatorname{tg} a \frac{\partial}{\partial \Omega_1} + \frac{1}{\cos a}\frac{\partial}{\partial \Omega_2}\right)$$

We introduced here the notation

$$b_{ik}^{(\ell m)} = \frac{1}{2}\left(\ell_i^{(\ell)}\, \ell_k^{(m)} + \ell_i^{(m)}\, \ell_k^{(\ell)}\right) \qquad /2.44/$$

For the sake of completeness let's write down the commutation relations

$$\left[B_{ik}, B_{j\ell}\right] = \frac{i}{2}\left(J_{i\ell}\, \delta_{kj} - J_{jk}\, \delta_{i\ell}\right) + \frac{i}{2}\left(J_{ij}\, \delta_{k\ell} - J_{\ell k}\, \delta_{ij}\right) \qquad /2.45/$$

$$\left[B_{ik}, J_{j\ell}\right] = \frac{i}{2}\left(B_{i\ell}\, \delta_{kj} - B_{jk}\, \delta_{i\ell}\right) - \frac{i}{2}\left(B_{ij}\, \delta_{k\ell} - B_{\ell k}\, \delta_{ij}\right) \qquad /2.46/$$

2.4. The Cubic Operator $\hat{\Omega}$

Finally we calculate the operator $\hat{\Omega}$. Let's introduce the operators H_+ and H_- - which are the usual SU (2) raising and lowering operators taken at the value of the Euler angle $-2\Omega_3 = 2\varphi_1 = 0$

$$H_{\pm} = \frac{1}{\sqrt{2}}\left[\frac{\partial}{\partial a} \pm i\frac{1}{\sin a}\frac{\partial}{\partial\lambda} \pm \frac{i}{2}\,\mathrm{ctg}\,a\,\frac{\partial}{\partial\Omega_3}\right] \qquad /2.47/$$

and the operators $\frac{\partial}{\partial\Omega_+}$ and $\frac{\partial}{\partial\Omega_-}$:

$$\frac{\partial}{\partial\Omega_{\pm}} = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial\Omega_1} \pm i\,\frac{\partial}{\partial\Omega_2}\right) = \frac{1}{\sqrt{2}}\,e^{\mp i\varphi_1}\left[i\,\frac{\partial}{\partial\Theta} \mp \frac{1}{\sin\Theta}\frac{\partial}{\partial\varphi_2} \pm \mathrm{ctg}\Theta\frac{\partial}{\partial\varphi_1}\right] \qquad /2.48/$$

Using these notations, we obtain $\hat{\Omega}$ in the form

$$\hat{\Omega} = \sum_{i,j,k} J_{ij}\,J_{jk}\,B_{ki} =$$

$$= -\frac{1}{4}\left\{\sqrt{2}\left(\frac{\partial^2}{\partial\Omega_+^2}H_+ + \frac{\partial^2}{\partial\Omega_-^2}H_-\right) + \frac{\partial^2}{\partial\Omega_3^2}\frac{\partial}{\partial\lambda} + \right.$$

$$+ \Delta_\Theta\frac{\partial}{\partial\lambda} - \frac{1}{\cos a}\left(\Delta_\Theta - \frac{\partial^2}{\partial\Omega_3^2} + \frac{1}{2}\right)\frac{\partial}{\partial\Omega_3} +$$

$$\left. + \mathrm{tg}\,a\left[i\left(\frac{\partial^2}{\partial\Omega_+^2} - \frac{\partial^2}{\partial\Omega_-^2}\right)\frac{\partial}{\partial\Omega_3} - \frac{3}{2}\left(\frac{\partial^2}{\partial\Omega_+^2} + \frac{\partial^2}{\partial\Omega_-^2}\right)\right]\right\} \qquad /2.49/$$

The operator $\hat{\Omega}$ has a simple meaning in the classical approximation. Substituting velocities for derivatives and introducing $\dot{\xi} = p$ and $\dot{\eta} = q$, we can write

$$\frac{1}{2}\,\hat{\Omega} = (\xi J)(qJ) - (\eta J)(pJ) \qquad /2.50/$$

The derivative of this operator is obviously zero. If we now choose the $\vec{z}$ axis to be directed along J, and introduce two vectors in the space of permutations

$$\vec{x} = \begin{pmatrix}\xi_z\\ \eta_z\end{pmatrix} \quad \text{and} \quad \vec{y} = \begin{pmatrix}p_z\\ q_z\end{pmatrix} \qquad /2.51/$$

then (2,50) can be rewritten as

$$\frac{1}{2}\,\hat{\Omega} = (\vec{x}\times\vec{y})_3 \qquad /2.52/$$

The operator has the form of the third component of the angular momentum in the space of permutations. So the symmetry properties of the problem become clear: we deal with spherical symmetry in the coordinate space, and with axial symmetry in the space of permutations.

Before writing down the eigenfunctions of the obtained five operators, we have to make a few remarks. One can show, that $\hat{\Omega}$ is not necessary at small K-values, at which the degree of degeneracy is small. Indeed, the number of states at given K and ν values is defined by the usual SU (3) formula

$$n(K,\nu) = \frac{1}{8}(K + 2)(K + 2 - 2\nu)(K + 2 + 2\nu) \qquad /2.53/$$

Summing over 2ν from - K to K, we obtain the well-known expression

$$n(K) = \frac{1}{12}(K + 3)(K + 2)^2(K + 1) \qquad /2.54/$$

Maximal degeneracy occurs in the case of states with $\nu = 0$ at even K values, and $\nu = 1/2$ at odd K values.

$$\left.\begin{aligned} n(K,0) &= \frac{1}{8}(K + 2)^3 && K \text{ - even} \\ n(K, 1/2) &= \frac{1}{8}(K + 1)(K + 2)(K + 3) && K \text{ - odd} \end{aligned}\right\} \qquad /2.55/$$

On the other hand, at given K values there are $(K + 1)^2$ states with different J and M, since $J \in (0, K)$ and $M \in (-J, J)$. For $K \geqslant 4$ we have

$$n(K, 0) > (K + 1)^2$$

and /2.56/

$$n(K, 1/2) > (K + 1)^2$$

Thus, for $K < 4$ all states /with given K, J, M and ν values/ are simple; the fifth quantum number is not necessary. In the interval $4 \leqslant K < 8$ doubly degenerate states show up; in these cases the orthogonalization can be carried out simply by constructing symmetric and antisymmetric combinations. Only at $K = 8$ states with three-fold degeneracy appear, for which the orthogonalization requires more complicated calculations. Besides, states with $J = 0$ and $J = K$ values are not degenerate. Consequently, in fact for practical purposes it is enough to deal with four quantum numbers.

The number of states at given K and ν values is given in the Appendix. There it is shown in detail, that n-fold degeneracy appears at $K = 4n$.

2.5. Eigenfunctions

Finally, let's look for the harmonic functions ϕ , which fulfill the eigenvalue equations of the Laplace operator and the operator N with eigenvalues $K(K+4)$ and ν respectively. The general form is the following*

$$\phi^J_{M,\nu} = \sum_\kappa \sum_\mu a_\nu(\kappa,\mu)\, D^{K/2-\kappa}_{\nu,\frac{\mu}{2}}(\lambda,a,0)\, D^J_{\mu,M}(\varphi_1\theta\varphi_2) \qquad /2.57/$$

It is easy to understand the meaning of this solution. One can consider the second D-function - which is the eigenfunction of J^2 and J_3 - as an eigenfunction of a rotating rigid top with the projection of the angular momentum onto the moving axis, equal to μ . This projection is not conserved in our case, that's why we have to take a sum over different values of μ . That is just the point where an additional operator is needed to orthogonalize the obtained functions.

The coefficients $a_\nu(\kappa,\mu)$ have to be defined from the eigenvalue equation of the Laplacian /2,13/ and from

$$\hat{\Omega}\phi^J_{M,\nu} = \Omega\phi^J_{M,\nu} \qquad /2.58/$$

These equations are unfortunately somewhat complicated:

$$\begin{aligned}\sum_\kappa \sum_\mu \Bigg\{ \Bigg[& a_\nu(\kappa,\mu-2)\frac{i}{2}\sqrt{\left(\frac{K}{2}-\frac{\mu}{2}-\kappa+1\right)\left(\frac{K}{2}+\frac{\mu}{2}-\kappa\right)}\sqrt{(J-\mu+2)(J-\mu+1)(J+\mu-1)(J+\mu)} + \\ & + a_\nu(\kappa,\mu+2)\frac{i}{2}\sqrt{\left(\frac{K}{2}+\frac{\mu}{2}-\kappa+1\right)\left(\frac{K}{2}-\frac{\mu}{2}-\kappa\right)}\sqrt{(J+\mu+2)(J+\mu+1)(J-\mu-1)(J-\mu)} + \\ & + a_\nu(\kappa,\mu)\left(i\nu\mu^2 + i\nu J(J+1) + 4\Omega\right)\Bigg]\, D^{K/2-\kappa}_{\nu,\frac{\mu}{2}}(\lambda,a,0)\, D^J_{\mu,M}(\varphi_1\theta\varphi_2) + \\ & + a_\nu(\kappa,\mu)\Bigg[\frac{i\mu}{\cos a}\left(J(J+1)-\mu^2+\frac{1}{2}\right) D^{K/2-\kappa}_{\nu,\frac{\mu}{2}}(\lambda,a,0)\, D^J_{\mu,M}(\varphi_1\theta\varphi_2) + \qquad /2.59/ \\ & + \operatorname{tg} a\Bigg(-\left(\frac{\mu}{2}+\frac{3}{4}\right)\sqrt{(J-\mu)(J+\mu+1)(J-\mu-1)(J+\mu+2)}\, D^{K/2-\kappa}_{\nu,\frac{\mu}{2}}(\lambda,a,0)\, D^J_{\mu+2,M}(\varphi_1\theta\varphi_2) + \\ & + \left(\frac{\mu}{2}-\frac{3}{4}\right)\sqrt{(J+\mu)(J-\mu+1)(J+\mu-1)(J-\mu+2)}\, D^{K/2-\kappa}_{\nu,\frac{\mu}{2}}(\lambda,a,0)\, D^J_{\mu-2,M}(\varphi_1\theta\varphi_2)\Bigg)\Bigg]\Bigg\} = 0\end{aligned}$$

---

* The solution is given in a similar form in [15] .

and

$$\sum_{\kappa}\sum_{\mu}\Bigg\{a_\nu(\kappa,\mu)\Big[\frac{K}{2}\Big(\frac{K}{2}+2\Big)-\Big(\frac{K}{2}-\kappa\Big)\Big(\frac{K}{2}-\kappa+1\Big)-\frac{\mu}{2}+\frac{\nu}{\cos a}-$$

$$-\frac{1}{2\cos^2 a}J(J+1)+\frac{\mu^2}{2\cos a}\Big]D^{K/2-\kappa}_{\nu,\frac{\mu}{2}}(\lambda,a,0)\,D^J_{\mu,M}(\varphi_1\theta\varphi_2)-$$

$$-a_\nu(\kappa,\mu-2)\,i\,\mathrm{tg}\,a\sqrt{\Big(\frac{K}{2}-\frac{\mu}{2}-\kappa+1\Big)\Big(\frac{K}{2}+\frac{\mu}{2}-\kappa\Big)}\,D^{K/2-\kappa}_{\nu,\frac{\mu}{2}}(\lambda,a,0)\,D^J_{\mu-2,M}(\varphi_1\theta\varphi_2)+$$

$$+a_\nu(\kappa,\mu)\Big[-\frac{i\sin a}{4\cos^2 a}\sqrt{(J-\mu)(J+\mu+1)(J-\mu-1)(J+\mu+2)}\,D^{K/2-\kappa}_{\nu,\frac{\mu}{2}}(\lambda,a,0)\,D^J_{\mu+2,M}(\varphi_1\theta\varphi_2)+$$

$$+\frac{i\sin a}{4\cos^2 a}\sqrt{(J+\mu)(J-\mu+1)(J+\mu-1)(J-\mu+2)}\,D^{K/2-\kappa}_{\nu,\frac{\mu}{2}}(\lambda,a,0)\,D^J_{\mu-2,M}(\varphi_1\theta\varphi_2)\Big]\Bigg\}=0$$

/2.60/

Although it is quite easy to solve this set of equations for every particular case, we couldn't obtain so far a general solution.

The practical calculations are getting simpler, if we take into account some properties of the eigenfunctions. Note, that in /2,57/ the D-functions corresponding to the spacial rotations form themselves an orthonormal set. Consequently, the eigenfunctions $\phi^J_{M,\nu}$ have to be orthogonal in the space of a and λ at any given values of the Euler angles $\varphi_1, \theta, \varphi_2$. We can therefore put $\varphi_1=\theta=\varphi_2=0$ after we have applied Ω. In other words, the problem reduces to the orthogonalization of the functions of a and λ.

If we introduce the operators

$$J_\pm=\frac{1}{\sqrt{2}}\big(J_1\pm iJ_2\big)$$

/2.61/

$$J_o=-iJ_3$$

which act on $D^J_{\mu M}(\varphi_1\,\theta\,\varphi_2)$ in the following way:

$$J_{\pm} D^J_{\mu M}(\varphi_1 \theta \varphi_2) = \mp \frac{i}{\sqrt{2}} \sqrt{(J \mp M)(J \pm M+1)}\; D^J_{\mu, M\pm 1}(\varphi_1 \theta \varphi_2)$$

$$J_o D^J_{\mu M}(\varphi_1 \theta \varphi_2) = iM\, D^J_{\mu M}(\varphi_1 \theta \varphi_2) \qquad /2.62/$$

then $\hat{\Omega}$ can be rewritten in the form

$$\hat{\Omega} = \sum_{i,j,k} B_{ij} J_{jk} J_{ki} = \frac{1}{4}\Big\{ B_{--} J_+^2 + B_{++} J_-^2 - B_{+-}\left(J_+J_- + J_-J_+ - 2J_o^2\right) -$$

$$-B_{-o}\left(J_+J_o + J_oJ_+\right) - B_{+o}\left(J_-J_o + J_oJ_-\right)\Big\} =$$

$$= \frac{1}{4}\Big\{ \sqrt{2}\left(H_-J_+^2 + H_+J_-^2\right) + \frac{\partial}{\partial\lambda}\left(-J^2 - J_o^2\right) - \qquad /2.63/$$

$$- \frac{i}{2\cos a}\left[\left(\frac{\partial}{\partial\Omega_+} + i \sin a \frac{\partial}{\partial\Omega_-}\right)\left(J_+J_o + J_oJ_+\right) + \left(\frac{\partial}{\partial\Omega_-} - i \sin a \frac{\partial}{\partial\Omega_+}\right)\left(J_-J_o + J_oJ_-\right)\right]\Big\}$$

Taking into account $D^J_{\mu M}(0,0,0) = \delta_{\mu M}$ we obtain the equations

$$\sum_{\kappa}\Big\{ \frac{i}{2} a_\nu(\kappa,2) \sqrt{\left(\frac{K}{2} - \kappa\right)\left(\frac{K}{2} - \kappa + 1\right)} \sqrt{J(J+1)(J-1)(J+2)}\; D^{K/2-\kappa}_{\nu,o}(\lambda,a,0) +$$

$$+ \frac{i}{2} a_\nu(\kappa,-2) \sqrt{\left(\frac{K}{2} - \kappa\right)\left(\frac{K}{2} - \kappa + 1\right)} \sqrt{J(J+1)(J-1)(J+2)}\; D^{K/2-\kappa}_{\nu,o}(\lambda,a,0) +$$

$$+ \left(i\nu J(J+1) + 4\Omega\right) a_\nu(\kappa,0)\, D^{K/2-\kappa}_{\nu,o}(\lambda,a,0) +$$

$$+ \frac{\sin a}{4\cos a} \sqrt{J(J+1)(J-1)(J+2)}\left(a_\nu(\kappa,2)\, D^{K/2-\kappa}_{\nu,1}(\lambda,a,0) + a_\nu(\kappa,-2)\, D^{K/2-\kappa}_{\nu,-1}(\lambda,a,0)\right)\Big\} = 0$$

$$/2.64/$$

for even K values, (fixing $M = 0$), and

$$\sum_{\kappa}\Big\{ \frac{i}{2} a_\nu(\kappa,3) \sqrt{\left(\frac{K}{2} - \kappa - \frac{1}{2}\right)\left(\frac{K}{2} - \kappa + \frac{3}{2}\right)} \sqrt{(J-1)(J+2)(J-2)(J+3)}\; D^{K/2-\kappa}_{\nu,1/2}(\lambda,a,0) +$$

$$+ \frac{i}{2} a_\nu(\kappa,-1)\left(\frac{K}{2} - \kappa + \frac{1}{2}\right) J(J+1)\, D^{K/2-\kappa}_{\nu,1/2}(\lambda,a,0) +$$

$$+ia_\nu(\kappa,1)\left[\nu\left(J(J+1)+1\right)+\frac{1}{4\cos a}\,3(J-1)(J+2)+\frac{1}{4\cos a}\,J(J+1)-4i\Omega\right]D^{K/2-\kappa}_{\nu,1/2}(\lambda,a,0)+$$

$$+\frac{1}{4}\frac{\sin a}{\cos a}\,a_\nu(\kappa,-1)\;J(J+1)\;D^{K/2-\kappa}_{\nu,-1/2}\,(\lambda,a,0)\;-$$

$$-\frac{1}{4}\frac{\sin a}{\cos a}\,a_\nu(\kappa,3)\;.\;3\sqrt{(J-1)(J+2)(J+3)(J-2)}\,D^{K/2-\kappa}_{\nu,3/2}\,(\lambda,a,0)\Big\}=0 \qquad /2.65/$$

for odd K values (and $M = 1$) respectively.

III. ANOTHER WAY OF CONSTRUCTION OF A SET OF EIGENFUNCTIONS

There is another way to find a complete set of eigenfunctions for the three-body problem. In fact, the problem becomes complicated because of the requirement of definite permutation symmetry properties. Without them it would be simple to construct the wanted functions with help of the graphical method of the so-called "tree-functions", which was proposed by Vilenkin and Smorodinsky [17] . We have to modificate these functions, i.e. we have to find a transformation from the complete set of "tree-functions" to the K-harmonics. /K-harmonics are hyperspherical functions possessing definite symmetry properties with respect to the permutations; they were introduced first by Simonov and Badalyan [3]/. Thus we construct the "tree-functions" which are the eigenfunctions of the Laplacian, and are characterized by the quantum numbers

$$K,\ j_1,\ M_1,\ j_2,\ M_2 \qquad /3.1/$$

where j_1M_1, j_2M_2 are the angular momenta and their projections conjugated to ξ and η . We have to transform these functions first to a set with given total angular momentum, that is, to a set characterized by

$$K,\ J,\ M,\ j_1,\ j_2 \qquad /3.2/$$

In the next step we pass over to the quantum numbers

$$K,\ J,\ M,\ \nu,\ (j_1\ j_2) \qquad /3.3/$$

corresponding to the K-harmonics. In order to do that it is necessary to carry out a simple Fourier transform. To be correct, $(j_1 j_2)$ is not a real quantum number in the sense, that functions corresponding to different pairs $(j_1 j_2)$ do not form an orthonormal set, but this notation demonstrates where we got these functions from. Let's point out, that j_1 and j_2 cease to be eigenvalues any more after performing the Fourier transform.

The calculation of the explicit form of the functions corresponding to /3,3/ is given in details in [12] - [13] . Here we present only the final expression.

$$\phi^{j_1 j_2}_{JM\nu}(\vec{\xi}\cdot\vec{\eta}) = A_{JM} \sum_m \sum_{\mu,\delta} \sum_\kappa \left(j_1, \frac{\mu+\delta}{2}; j_2, \frac{\mu-\delta}{2}\middle| J; \mu\right)^2 .$$

$$\cdot \frac{\left(\frac{K-\delta}{4}, W+\frac{\mu}{4}; \frac{K+\delta}{4}, -W+\frac{\mu}{4}\middle|\frac{K}{2}-\kappa; \frac{\mu}{2}\right)(-1)^{\frac{K+\mu-\delta}{4}-\frac{\nu}{2}+\kappa}}{\left(\frac{j_1}{2}+m, \frac{\mu+\delta}{4}; \frac{j_2}{2}+n-m, \frac{\mu-\delta}{4}\middle|\frac{K}{2}; \frac{\mu}{2}\right) 2^{K/4}} .$$

$$\frac{\Delta^{(J)}_{o\mu}\,\Delta^{\left(\frac{K}{2}-\kappa\right)}_{\delta/2,\nu}}{\Delta^{\left(\frac{K}{2}\right)}_{\frac{K}{2},\frac{\mu}{2}}} \left[\frac{(j_1+2m)!\,(j_2+2n-2m)!}{(K+\kappa+1)!\,\kappa!}\right]^{1/2} \begin{pmatrix} n+j_1+\frac{1}{2} \\ m \end{pmatrix} \begin{pmatrix} n+j_2+\frac{1}{2} \\ n-m \end{pmatrix}$$

$$\cdot\, D^{K/2-\kappa}_{\nu,\mu/2}(\lambda,a,0)\, D^J_{\mu M}(\varphi_1 \theta \varphi_2) \qquad /3.4/$$

where

$$A_{JM} = \frac{(-1)^{\frac{M-j_2}{2}}}{2^{\frac{j_1+j_2}{2}}} \frac{4\pi}{(2J+1)^{1/2}} \frac{1}{\left[N^{j_1+1/2,\, j_2+1/2}_{\frac{K-j_1-j_2}{2}}\right]^{1/2}} \left[\frac{(j_1+M_1)!\,(j_2+M_2)!}{(j_1-M_1)!\,(j_2-M_2)!}\right]^{1/2} .$$

$$\cdot\left(j_1, 0; j_2, 0\middle|J;0\right)\left(j_1, M_1; j_2, M_2\middle|J;M\right) \qquad /3.5/$$

$$N^{j_1+\frac{1}{2},\, j_2+\frac{1}{2}}_{\frac{K-j_1-j_2}{2}} = -\frac{1}{2} \frac{\Gamma\left(\frac{K}{2}-\frac{j_1}{2}+\frac{j_2}{2}+\frac{3}{2}\right)\Gamma\left(\frac{K}{2}+\frac{j_1}{2}-\frac{j_2}{2}+\frac{3}{2}\right)}{(K+2)\,\Gamma\left(\frac{K}{2}-\frac{j_1+j_2}{2}+1\right)\Gamma\left(\frac{K}{2}+\frac{j_1+j_2}{2}+2\right)} ;$$

$$\Delta^{(m)}_{k\ell} = D^m_{k\ell}\left(0, \frac{\pi}{2}, 0\right) = P^m_{k\ell}\left(\cos\frac{\pi}{2}\right) ; \qquad /3.6/$$

$$\delta = \mu_1 - \mu_2 \quad ; \qquad W = \frac{j_2-j_1}{4} + \frac{n}{2} - m$$

Comparing now the expressions /2,57/ and /3,4/ one can establish, that the general form of the solution /2,57/ was chosen in the right way. However, looking at the structure of the coefficient of $D^{K/2-\kappa}_{\nu,\mu/2}(\lambda,a,0)\, D^{J}_{\mu M}(\varphi_1 \theta \varphi_2)$ in /3,4/ it is easy to understand that our attempts to determine $a_\nu(\kappa,\mu)$ directly couldn't be successful. Still, now we can somewhat more precisely describe the method of determination of the $\hat{\Omega}$ eigenfunctions. The solutions of the eigenvalue equations for K and Ω have to be linear combinations of the functions /3,4/ :

$$\phi^{J}_{M,\nu} = \sum_{j_1 j_2} C_{(j_1 j_2)} \phi^{j_1 j_2}_{JM\nu} (\vec{\xi}, \vec{\eta}) \qquad /3.7/$$

where $j_1 j_2$ will run over each pair of values which can give the total angular momentum J such that $J \leq j_1 + j_2 \leq K$. And what is more: one can show, that in fact there is no need to take every possible pair of j_1, j_2. The number of the necessary pairs j_1 and j_2 in each sum /3,7/ is equal to the degree of degeneracy of the given state which is considered.

CONCLUSIONS

The problem of constructing a basis for a system of three free particles, realizing representations of the three-dimensional rotation group and of the permutation group, is quite simple in principle. To solve the problem, however, turned out to be rather hard. We calculated a set of equations for determining the eigenfunctions, but we could'nt get so far a general solution for it. /The obtained formulae are complicated, because the polynomials, which we are dealing with, are not classical and their theory is not worked out yet. If our method will lead to useful results, it will not be difficult to study the properties of these new polynomials and tabulate them./

Note, that, if one is dealing with a larger number of particles, then the formulae will be still more complicated; in a certain sense the situation is similar to the transition from hypergeometrical functions of one variable to those of few variables, the theory of which is almost not known.

We have pointed out, that for practical purposes the general solution of the problem is in fact not necessary, and there is no need to use the operator $\hat{\Omega}$. In spite of that we insist on deriving the solution in a closed form, the more so since the problem seems to be practically solved. As it was shown, the eigenvalue equations can be simplified considerably, and it remains only to calculate the coefficients and obtain numerical results. /We present them in our next paper./

There are several possibilities to apply the technics developped here. First of all, as soon as the quantum mechanical three-body problem, which we have delt with, has the same symmetry properties as the classical one, it was interesting to investigate the classical problem from this group-theoretical point of view [14] . The equations of motion were obtained very easily for both the case of free particles and of different potentials.

The classification of a three-body system presented in this paper can be used as well for the analysis of three-particle decay processes. Namely: dealing with a Dalitz plot for decay processes it seems to be useful to expand the point density inside the physical region into a series of orthonormal functions. /Such an expansion is similar to the usual phase analysis for two-particle decays, and it can be used for coding experimental data, for calculation of different correlation functions, etc./ One can choose for the set of basis functions our K-harmonics; this choice will be especially suitable when there will be an experimental possibility to notice correlation between the momenta of particles. The expansion procedure is worked out, but no numerical calculations are done yet.

From a practical point of view it is of course essential to develop a method to calculate matrix elements of pairwise interactions introducing different potentials. It will be necessary to obtain a proper approximation for bound states as well.

It would be also of interest to see, whether it is possible to make use of an expansion of that kind, which is described in the present paper, for the motion of a massive top. Especially interesting /and so far not well understood/ is the case of the Kovalevskaya top [18] , the quantum analogue of which is not known yet.

APPENDIX

States with given K values can be obtained in the usual way, by constructing tensors and pseudotensors from the SU (3) basis vectors z_i . These states are labelled by quantum numbers J and ν according to the chain SU (3) $\supset$ O (3) x O (2) . As an example, we list all possible states with given K values /in the interval $1 \leq K \leq 5$/ and different J and ν . The degeneracy appears clearly at K = 4. In order to get the number of states, it will be sufficient to consider instead of the polynomials itselves their first /highest/ terms, which will be denoted as P in the following.

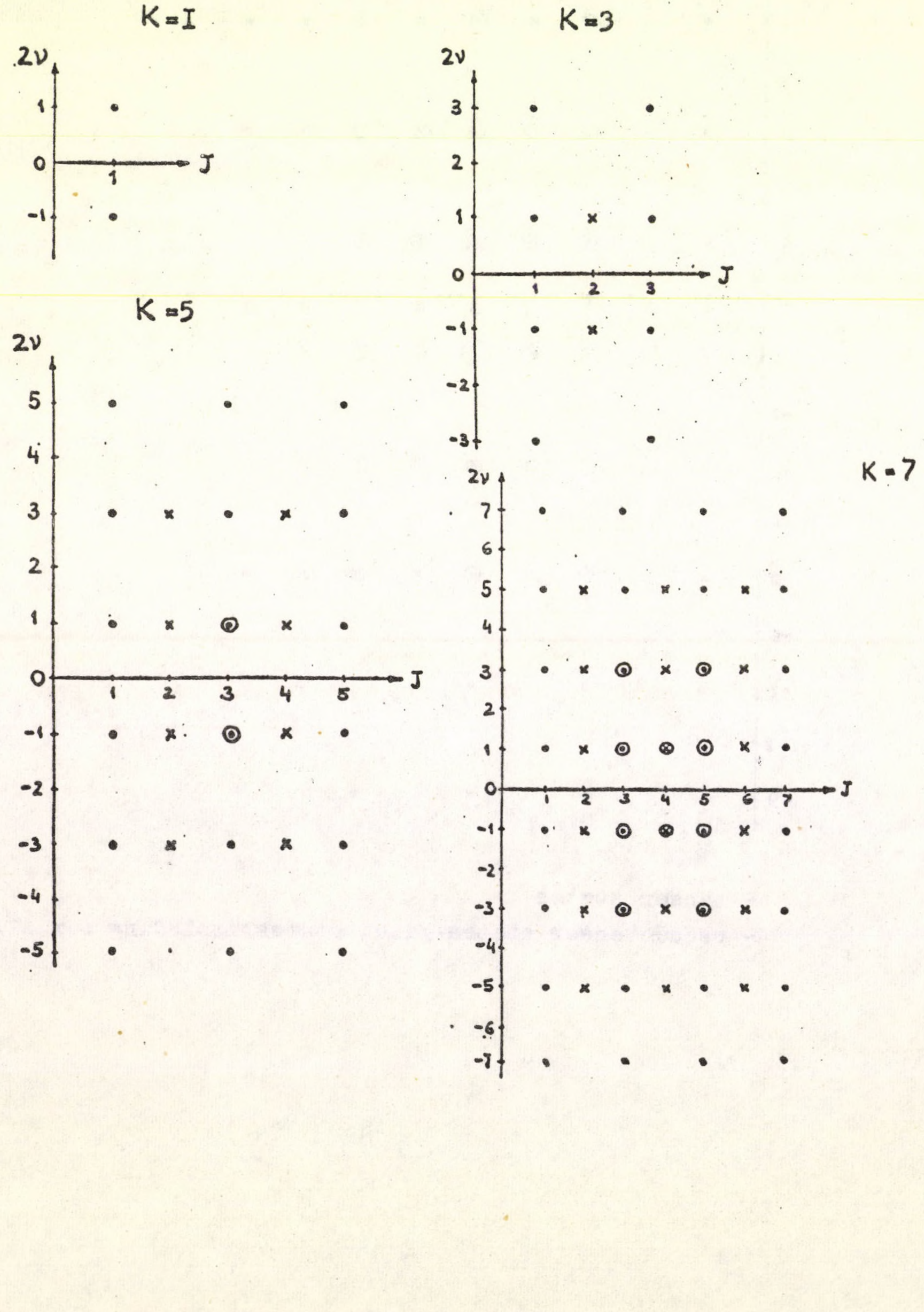
K	P	$\|2\nu\|$	J	n	$n(K,\pm\nu)$	n(K)
1	z_i	1	1	3×2	3×2	6
	z_i^*					
2	$z_i z_k$	2	2	5×2	6×2	20
	$z_i^* z_k^*$					
	z^2		0	1×2		
	z^{*2}					
	$z_i z_k^*$	0	2	5	8	
	$z \times z^*$		1	3		
3	$z_i z_j z_k$	3	3	7×2	10×2	50
	$z_i^* z_j^* z_k^*$					
	$z^2 z_k$		1	3×2		
	$z^{*2} z_k^*$					
	$z_i^* z_j z_k$	1	3	7×2	15×2	
	$z_i^* z_j^* z_k$					
	$z_i^* z^2$		1	3×2		
	$z^{*2} z_k$					
	$z_i (z \times z^*)$		2	5×2		
	$z_i^* (z \times z^*)$					

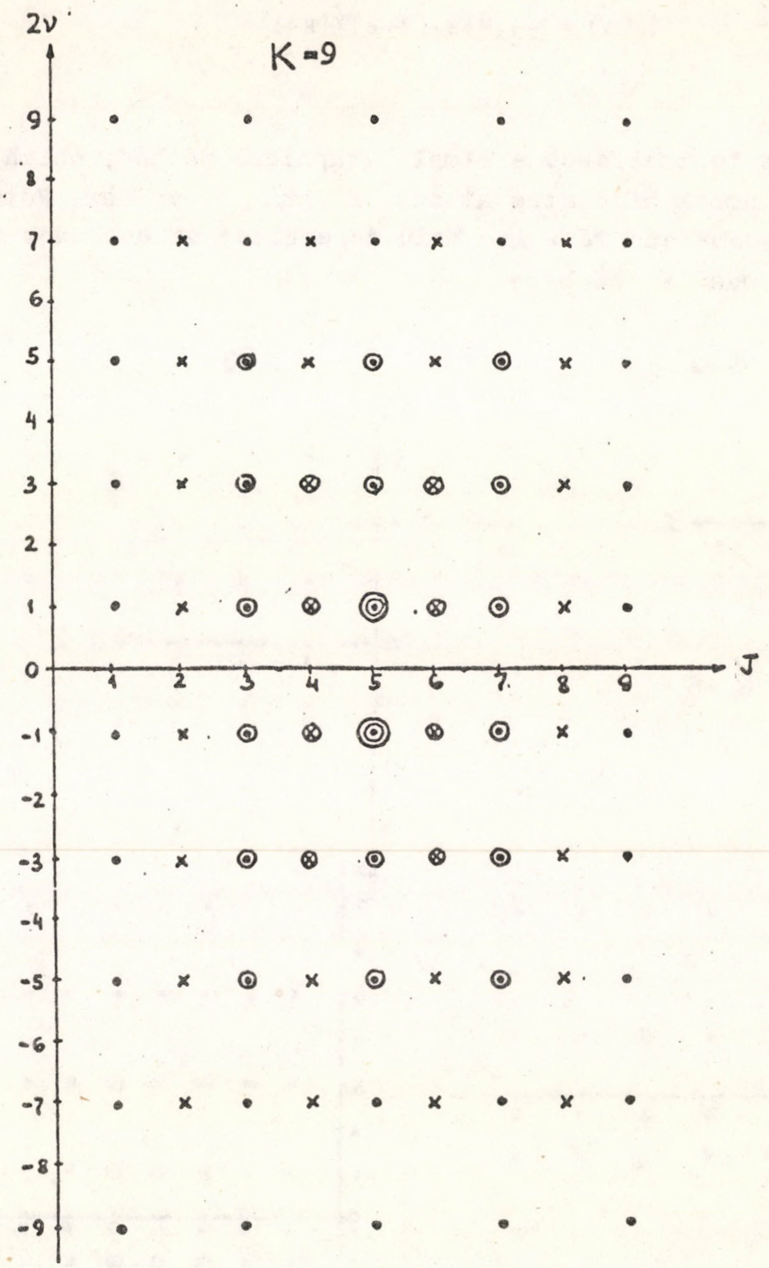
K	P	$\lvert 2\nu \rvert$	J	n	$n(K,\pm\nu)$	$n(K)$
	$z_i z_j z_k z_\ell$		4	9×2		
	$z_i z_k z^2$	4	2	5×2	15×2	
	z^4		0	1×2		
	and complex conjugates					
	$z_i z_j z_k z_\ell^*$		4	9×2		
	$z_i z^2 z_\ell^*$		2	5×2		
4	$z_i z_j (z\times z^*)$	2	3	7×2	24×2	105
	$z^2 (z\times z^*)$		1	3×2		
	and complex conjugates					
	$z_i z_j z_k^* z_\ell^*$		4	9		
	$z_i z_j z^{*2}$		2	5		
	$z^2 z_k^* z_\ell^*$	0	2	5	27	
	$z^2 z^{*2}$		0	1		
	$z_i z_k^* (z\times z^*)$		3	7		
	$z_i z_j z_k z_\ell z_m$		5	11×2		
	$z_i z_j z_k z^2$	5	3	7×2	21×2	
	$z_i z^4$		1	3×2		
	and complex conjugates					
	$z_i z_j z_k z_\ell z_m^*$		5	11×2		
	$z_i z_j z^2 z_m^*$		3	7×2		
5	$z^4 z_m^*$	3	1	3×2	35×2	
	$z_i z_j z_k (z\times z^*)$		4	9×2		196
	$z_i z^2 (z\times z^*)$		2	5×2		
	and complex conjugates					
	$z_i z_j z_k z_\ell^* z_m^*$		5	11×2		
	$z_i z^2 z_\ell^* z_m^*$		3	7×2		
	$z_i z_j z_k z^{*2}$		3	7×2		
	$z_i z^2 z^{*2}$	1	1	3×2	42×2	
	$z_i z_j (z\times z^*) z_m^*$		4	9×2		
	$z^2 (z\times z^*) z_m^*$		2	5×2		
	and complex conjugates					

$$N(K,\pm\nu) = \frac{1}{8}(K+2)(K+2-2\nu)(K+2+2\nu) \qquad /A1/$$

$$n(K) = \frac{1}{12}(K+3)(K+2)^2(K+1) \qquad /A2/$$

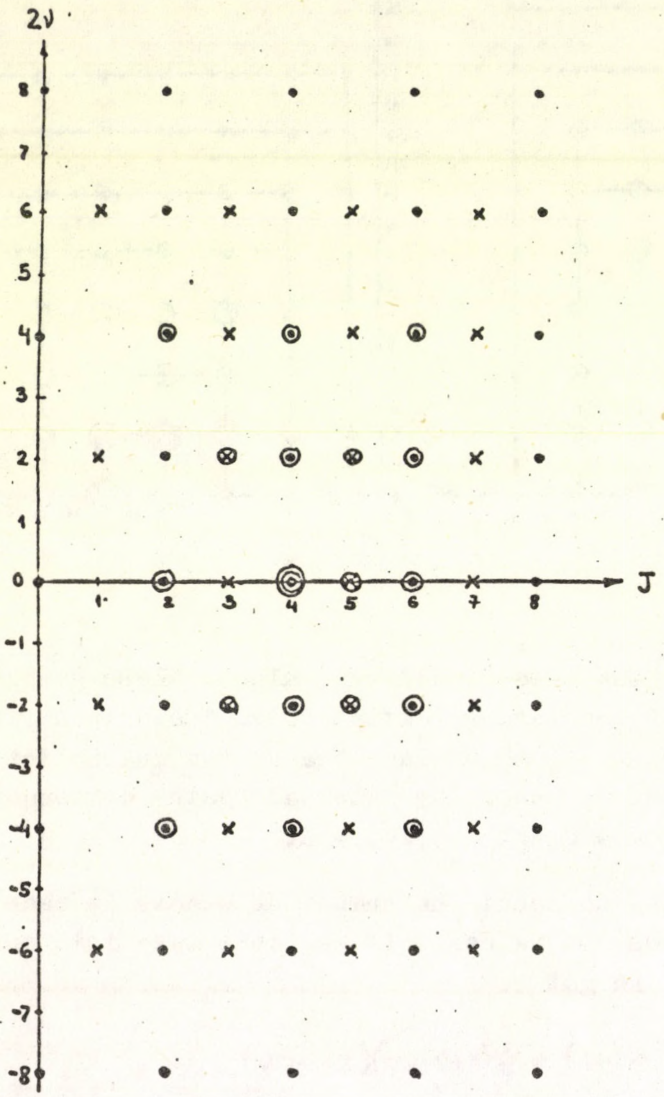
It is worthwile to introduce a simple graphical method, which enables us to obtain the number of states at any K and ν values. Note, that we do not mark the obvious 2J + 1 - fold degeneracy of each dot on the graphs. In the case of odd K we have



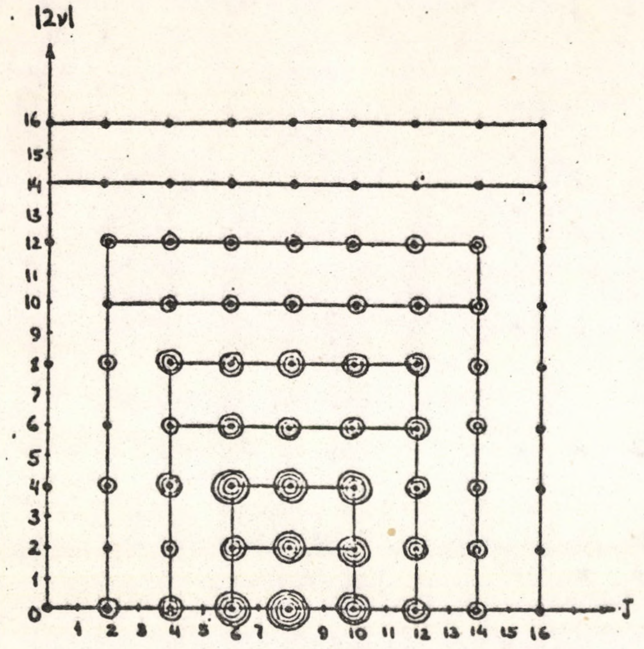


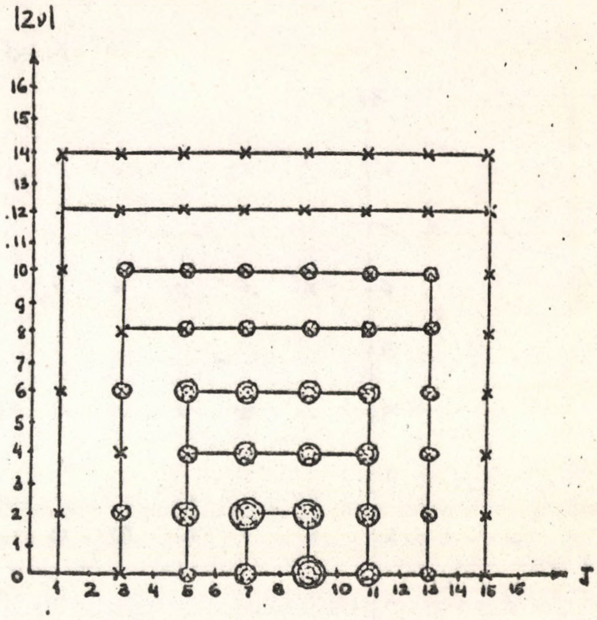
- tensor states
- pseudotensor states /i.e. states including z x z*/.

K =8



It can be seen, that again the graphs can be constructed from two elementary graphs; for example, in the case K = 16 we have





Analogously to the case of odd K values, these graphs are put together from "gates" of increasing multiplicity. The only difference is, that in the left-hand side column of each "gate" the multiplicity of every second state is decreased by one. The external "gate" corresponds to the quantum numbers J = 0, J = K and $|2\nu| = K$.

It is quite easy to count the number of states in each row of the graphs /i.e. at given ν values/. If we don't take into account the different values of M, we get

$$
n'(K,S) = \begin{cases} \frac{1}{2}(S+1)(K-S+1) = \frac{1}{8}(K+2+2\nu)(K+2-2\nu) \\ \text{in the case of odd } K \text{ values, and for} \\ \text{even } K \text{ and odd } S \text{ values;} \\ \\ \frac{1}{2}(S+1)(K-S+1) + \frac{1}{2} = \frac{1}{8}(K+2+2\nu)(K+2-2\nu) + \frac{1}{2} \\ \\ \text{for even } K \text{ and even } S \text{ values} \end{cases} \qquad /A3/
$$

where

$$S = \frac{2\nu + K}{2} \ .$$

The number of states on the graph which can be obtained by summing up the rows /that means, summing over ν / is the following:

$$n'(K) = \begin{cases} \frac{1}{12}(K+2)(K^2+4K+6) & \text{for odd } K \\ \frac{1}{12}(K+1)(K^2+5K+6) & \text{for even } K \end{cases} \qquad /A4/$$

Note, that the number of all states we get if we take into account, that the multiplicity of a state with a given J value is 2J+1. Thus the formulae /A4/ present the number of states with given K, J and ν values, independently of the value of M.

Finally, it is interesting to compare /A3/ with /A1/ . From this comparison it follows, that the "average" number of states at a given K value is equal to K + 1.

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REFERENCES

[1] W. Zickendraht, Ann. Phys. /N.Y./ 35 18 /1965/
[2] W. Zickendraht, Phys. Rev. 159 1448 /1967/
[3] А. М. Бадалян, Ю.А. Симонов, ЯФ 3 6 /1966/
[4] Ю.А. Симонов, ЯФ 3 630 /1966/
[5] В.В. Пустовалов, Ю.А. Симонов ЖЭТФ 51, 345 /1966/
[6] A.K. Bhatia, A. Temkin, Rev.Mod. Phys. 36, 1050 /1964/
[7] F.T. Smith, Journ.Math.Phys. 3 735 /1962/
[8] A.J. Dragt, Journ.Math.Phys. 6 533 /1965/
[9] J.M. Lévy-Leblond, M. Lévy-Nahas, Journ.Math.Phys. 6 1571 /1965/
[10] J.Nyiri, Ya.A. Smorodinsky, Preprint E4-4043, JINR Dubna /1968/
[11] Ю. Нири, Я.А. Смородинский, ЯФ 9 882 /1969/
[12] J.Nyiri, Ya.A. Smorodinsky, Preprint E2-4809 JINR Dubna /1969/
[13] Ю. Нири, Я.А. Смородинский, ЯФ 12 202 /1970/
[14] J.Nyiri, Ya.A.Smorodinsky, Preprint E2-5067 JINR Dubna /1970/
[15] R.C. Whitten, F.T. Smith, Journ.Math.Phys. 9 1103 /1968/
[16] G. Racah. Rev.Mod.Phys. 21 494 /1949/
[17] Н. Я. Виленкин, Г.И. Кузнецов, Я.А. Смородинский ЯФ 2 906 /1965/

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