KFKI-71-60

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Hungarian Academy of Sciences

CENTRAL RESEARCH INSTITUTE FOR PHYSICS

BUDAPEST

2017

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ABSTRACT

The expansion of the unequal-mass scattering amplitude in terms of Poincaré-group representations was considered for positive and zero values of s, the squared total four-momentum. The usual singularity problem at s=0 was avoidable, but it turned out, that the relevant variable is not j, the total angular momentum, but a quantity non-singularly related to the Poincaré-invariant $W_\mu W^\mu$ even at s=0. The notion of complex angular momentum and signature was reexamined, and some modification of the old formalism seemed useful. The results are perfectly compatible with dispersion relations and with the requirements of Regge behaviour. In the Appendix a theorem is proved for the expansion of a class of not square-integrable, but Regge behaved functions with respect to unitary E(2) representations /that is, for Fourier-Bessel expansions/.

РЕЗЮМЕ

Рассматривается разложение амплитуды рассеяния частиц неравных масс по представлениям группы Пуанкаре, при положительном и нулевом значениях квадрата полного момента s. Показано, что обычную особенность при s = 0 можно обойти, но выяснилось, что при этом основной переменной является не полный угловой момент j, а другая величина, которая связана с инвариантом $W_\mu W^\mu$ группы Пуанкаре даже при s = 0 неособенным образом. Пересмотрены понятия комплексных угловых моментов, а также сигнатуры и некоторые изменения в обычном формализме оказались полезными. Полученные результаты согласуются с требованиями дисперсионных соотношений и поведением Редже. В приложении доказана теорема, касающаяся разложения не квадратично интегрируемых функций, но показывающих поведение Редже по унитарным представлениям группы E(2).

KIVONAT

A nem egyenlő tömegü szórási amplitudó Poincaré-csoport ábrázolások szerinti sorfejtéseit vizsgáltuk a teljes négyes-impulzus négyzetének, s, pozitiv és nulla értékeire. A szokásos szingularitási probléma elkerülhető volt, de kiderül, hogy a lényeges változó nem j, a teljes impulzus momentum, hanem egy, a $W_\mu W^\mu$ Poincaré invariánssal s=0 -nál sem szinguláris kapcsolatban álló változó. Megvizsgáltuk a komplex impulzus momentum és a szignatura fogalmát, és a régi formalizmus néhány módositását hasznosnak találtuk. Eredményeink összhangban vannak a diszperziós relációk és a Regge-viselkedés követelményeivel. Az Appendix-ben egy tételt bizonyitunk nem négyzetesen integrálható, de Regge-viselkedést mutató függvények uniter E(2) ábrázolások szerinti sorfejtésére /azaz, Fourier-Bessel kifejtésére/.

INTRODUCTION

The difficulties of Regge pole theory at zero energy in the case of unequal-mass scattering have inspired many authors, and many different approaches have been proposed to solve the problem. The general attitude is to take for granted the presence of unpleasant singularities in the Watson-Sommerfeld transformed form of the unequal-mass scattering amplitude, and the task is just how to remove the singularities. On the other hand, one must realise, that even the presence of these singularities is questionable. What actually happens in the reggeization procedure is that some formulas, well-defined in the s-channel, are extrapolated to new regions, into the t or u-channel. It is far not trivial that, although the starting situation is very similar, everything must be learned from the equal-mass case. Instead, probably Fourier-analysis on Poincaré-group is the "magic word" one is to remember in the reggeization procedure.

Many authors have investigated the connection between the forms of the scattering amplitude obtained by Watson-Sommerfeld transformation and from direct group-theoretic expansions, mostly for space-like total four-momentum, $s < 0$ [1,2,3,4]. The present paper is mainly devoted to the problems at $s = 0$ in the unequal-mass case. Some steps of our approach were made in [5] and [6], but our results go far beyond theirs.

We are going to deal both with the limit of the Watson-Sommerfeld transform to $s = 0$ and with the connection of this limit with the group-theoretical expansion in terms of light-like Poincaré-representation matrix elements. These investigations lead to the following conclusion: the appropriate variable at $s = 0$ is not j, but w, the eigenvalue of the Poincaré-invariant $W_\mu W^\mu$, W_μ being the Pauli-Lubanski operator. As is well-known, at $s = 0$ real positive values of w correspond to unitary Poincaré representations /infinite spin representations/, they are sufficient to expand a square-integrable scattering amplitude. Complex values of w

correspond to non-unitary representations, and a complex angular momentum theory is to be formulated in terms of functions of the complex variable w. Obviously, when s is not zero, one may equally use w or j. On the other hand, one can not provide a /Poincaré/ group-theoretical interpretation to a theory, which uses the variable j at $s = 0$. /Our way of looking at the problems with unequal-mass scattering is very strongly supported by R.Hermann's book entitled "Fourier analysis on groups and partial wave analysis" [7]. In other words our suggestion is that s and j are not the "most economical" variables to formulate a complex angular momentum theory, but s and w are. /Also Feldman and Matthews have suggested that the correct variable to be used is not j but w [6]. See also ref. [8]./ The undesirable singularity at $s = 0$ is a consequence only of the uneconomical choice of variables. /The analogue of this phenomenon is well-known in context of the singularities which arise when using the variables s and $\cos\Theta_s$ instead of the "most economical" pair s,t./ In arriving at this conclusion group-theoretical interpretability is only a hint, rather than a necessary condition.

In this paper the scattering of two spinless particles with masses m and μ /pion-nucleon-type kinematics/ will be examined. In Section 2. some remarks on Poincaré representations are presented /for a detailed discussion see ref. [9]/, which are of basic importance in the subsequent investigations. In Section 3. the Watson-Sommerfeld representation of the scattering amplitude is given, and our modifications of the complex angular momentum are described in comparison with the conventional treatments. In Section 4. the $s = 0$ limit is calculated, and in Section 5. a comparison is made between the Sommerfeld-Watson representation and the expansion with respect to Poincaré representation matrix elements. In Section 6. some details of our approach are discussed, and in two appendices mathematical statements made in the previous sections are proved.

2. REMARKS ON POINCARÉ REPRESENTATIONS

If one takes the standpoint, that the Regge-Watson-Sommerfeld representation of the scattering amplitude is nothing else, but essentially a group-theoretical expansion in terms of Poincaré representations /this is supported, e.g., by the fact, that resonances are classified by putting them on Regge trajectories/, then the $s = 0$ problem of unequal-mass scattering can be, at least in part, transferred to the representation theory of the Poincaré group. Namely, the question arises, if the representations of the Poincaré group can be described in such a form, that is continuous in the Casimir eigenvalue $P_\mu^2 = s$ at $s = 0$, when the four-momentum P_μ becomes light-like. This problem was thoroughly investigated in ref. [9], and we summarize its most important points here.

The Poincaré group has been represented on a sufficiently large function space, and explicit functions in this space could be found with the following properties:

1. They are eigenfunctions of the four-momentum, P_μ, with arbitrary real eigenvalues p_μ; of $W_\mu W^\mu$, with arbitrary complex eigenvalues $sj(j+1) = w^2 - \frac{1}{4}s$, where $s = p_\mu^2$; and, of W_o with eigenvalue $p\lambda$, where p is the magnitude of the three-momentum $\underline{p}$, λ is the helicity. That is, the functions with given s and w form an irreducible set for representing the Poincaré group in helicity basis.

2. They are continuous functions of the four-momentum, p_μ, consequently of s as well. Appropriate normalization is essential to achieve continuity at $s = 0$. /The point $p_\mu = 0$ is a very peculiar one [9], and is unimportant in this paper. Hereafter $s = 0$ will always be associated with light-like four-vectors./

When having been in possession of basis functions, representation matrix elements of the Poincaré group have been calculated. The result is of the following form:

$$\langle p_\mu, w, \lambda | (a,\Lambda) | p'_\mu, w', \lambda' \rangle = N(s,w,w')\, \delta^4\left(p_\mu - \Lambda p'_\mu\right) D^w_{\lambda\lambda'}(\varphi,\Theta,\psi)\, \exp\left(-ip_\mu a_\mu\right) \qquad /2.1/$$

where $N(s,w,w')$ is a continuous function of $p_\mu^2 = s$, when w and w' are fixed. The function $D^w_{\lambda\lambda'}$ denotes the familiar representation functions of the groups SU/2/, SU/1,1/ or E/2/ depending on whether s is positive, negative or zero, respectively [10]. /In the cases when $s \neq 0$, more conventionally the label j is used instead of w./ The Euler-angles φ, Θ, ψ in the $D^w_{\lambda\lambda'}$ function are functions of the six parameters of the homogeneous Lorentz group element Λ and of the four components of p_μ. The method to determine the functions $\varphi(\Lambda,p_\mu)$, $\Theta(\Lambda,p_\mu)$, $\psi(\Lambda,p_\mu)$ is well-known, they are the Euler-angles of the Wigner-rotation $L^{-1}_{p_\mu} \Lambda L_{\Lambda^{-1}p_\mu}$, where L_{p_μ} and $L_{\Lambda^{-1}p_\mu}$ are boosts, which transform the four-vector $/\sqrt{s},0,0,0/$ /in case $s > 0$/, or the one $/0,0,0,\sqrt{-s}/$ /in case $s < 0$/ to p_μ and $/\Lambda p/_\mu$, respectively. It can be checked again, that the functions $D^w_{\lambda\lambda'}(\varphi,\Theta,\psi)$ are continuous functions of the components of p_μ, when w is fixed. This might be surprising, since similar statement is not true for $\Theta(\Lambda,p_\mu)$. Namely, $\lim_{s\to 0}\Theta(\Lambda,p_\mu) \equiv 0$ /p_μ becomes light-like!/, independently on Λ. On the other hand, if we calculate the matrix element /2.1/ directly for light-like representations, /that is, also the Euler-angles of the "Wigner-rotation" $\hat{L}^{-1}_{p_\mu} \Lambda \hat{L}_{\Lambda p^{-1}_\mu}$ with boosts $\hat{L}_{p_\mu}$ and $\hat{L}_{\Lambda p^{-1}_\mu}$ transforming a four-vector $/p,0,0,p/$ to p_μ and $(\Lambda p)_\mu$, respectively./, we find that $\Theta\left(\Lambda,p_\mu(\in)0,\infty\right)$. This discrepancy can be very easily eliminated by reinterpreting the function $D^{w}_{\lambda\lambda'}$ in the following manner: it is the representation matrix element $D^{w}_{\lambda\lambda'}(\varphi^v,\Theta^v,\psi^v) \equiv \left(D^{w}_{\lambda\lambda'}(\varphi,\Theta,\psi)!\right)$ of the little-group of the four vector $/p_o,0,0,p/$, $p_o^2 - p^2 = p_\mu^2 = s$, the Euler-angles of which being those of $\tilde{L}^{-1}_{p_\mu} \Lambda \tilde{L}_{\Lambda p^{-1}_\mu p}$ where $\tilde{L}_{p_\mu}$ and $\tilde{L}_{\Lambda p^{-1}_\mu p_\mu}$ are boosts transforming the four-vector $/p_o,0,0,p/$ into p_μ and $(\Lambda p)_\mu$, respectively. It is easy to verify, that

$$\varphi^v \equiv \varphi , \qquad \psi^v \equiv \psi ,$$

$$\Theta^v = \left[\frac{|p_o| + p}{\left|\,|p_o| - p\,\right|}\right]^{1/2} \Theta , \qquad /2.2/$$

and Θ^v is now continuous function of s even at $s = 0$. /As for the representations of little-groups for four-vectors like $/p_o,0,0,p/$ see, for example, ref. [11]/.

The significance of choosing Euler-variables which are continuous functions of s becomes clear, when we come to the next relevant point, to the orthogonality relations of the matrix elements:

$$I_1 = \int d^4a \, d\mu(\Lambda) \langle p_\mu,w,\lambda|(a,\Lambda)|p'_\mu ,w',\lambda'\rangle \langle p''_\mu,w'',\lambda''|(a,\Lambda)|p'''_\mu,w''',\lambda'''\rangle^* , \qquad /2.3/$$

where the integration goes over the translation and the homogeneous Lorentz group part of the Poincaré group. /Concerning the measure $d\mu(\Lambda)$ on the Lorentz group see, e.g., ref. [10]./
After performing trivial integrations one obtains:

$$I_1 = N(s,w,w') \, N(s,w'',w''') \, \delta^4\left(p_\mu - (\Lambda p')_\mu\right) \delta^4\left(p''_\mu - (\Lambda p''')_\mu\right) \delta^4\left(p_\mu - p''_\mu\right) I_2 , \qquad /2.4/$$

where

$$I_2 = \int d\mu(\varphi^v,\Theta^v,\psi^v) \, D^{w}_{\lambda\lambda'}(\varphi^v,\Theta^v,\psi^v) \, D^{w''*}_{\lambda''\lambda'''}(\varphi^v,\Theta^v,\psi^v) =$$
$$= \left(|p_o| + p\right)^2 N(s,w,w'') \, \delta_{\lambda\lambda''} \delta_{\lambda'\lambda'''} . \qquad /2.5/$$

In the expression /2.5/ $d\mu(\varphi^V,\Theta^V,\psi^V)$ is the measure for the little-group of $/p_o,0,0,p/$. Lengthy, but straightforward calculation gives, e.g., for $s > 0$:

$$d\mu(\varphi^V,\Theta^V,\psi^V) = \left[\frac{|p_o| + p}{|p_o| - p}\right]^{1/2} \sin\left(\left[\frac{|p_o| - p}{|p_o| + p}\right]^{1/2} \Theta^V\right) d\varphi^V d\Theta^V d\psi^V =$$

$$= \frac{|p_o| + p}{|p_o| - p} d(\cos\Theta)\, d\varphi\, d\psi \equiv \frac{|p_o| + p}{|p_o| - p} d\mu(\varphi,\Theta,\psi) \qquad /2.6/$$

and

$$N(s,w,w") = \frac{1}{(2j+1)s} \delta_{jj"} \qquad /2.7/$$

where $w^2 = sj(j+1) + \frac{1}{4}s$, $w"^2 = sj"(j"+1) + \frac{1}{4}s$. We call attention to the fact, that in the integral /2.5/ the measure $d\mu(\varphi^V,\Theta^V,\psi^V)$ has appeared, rather than $d\mu(\varphi,\Theta,\psi)$. This is strongly correlated with the singular behaviour of the angle Θ at $s = 0$.

The formulas /2.5-7/ make possible to write down the partial wave expansion, that is, the expansion with respect to irreducible, time-like Poincaré representations for an unequal-mass scattering amplitude in such a form, which we expect, after reggeization, to have nice properties even at zero energy:

$$\langle p_3,s_3,\lambda_3;\ p_4,s_4,\lambda_4|T|p_1,s_1,\lambda_1;\ p_2,s_2,\lambda_2\rangle =$$

$$= (2\pi)^4\ \delta^4(p_1+p_2-p_3-p_4)\, F_{\lambda_1\lambda_2\lambda_3\lambda_4}(s,t) = \qquad /2.8/$$

$$= (2\pi)^4\ \delta^4(p_1+p_2-p_3-p_4) \frac{s}{(|p_o|+p)^2} \sum_j F_{\lambda_1\lambda_2\lambda_3\lambda_4}(s,j)(2j+1)\, d^j_{\lambda\mu}(\Theta_s) \quad ,$$

where $\lambda = \lambda_3 - \lambda_4$, $\mu = \lambda_1 - \lambda_2$, and Θ_s is the scattering angle in the center-of-mass /C.M./ frame for the s-channel. The partial wave amplitudes are defined as follows:

$$F_{\lambda_1\lambda_2\lambda_3\lambda_4}(s,j) = \frac{(|p_o|+p)^2}{s} \int_{-1}^{1} d(\cos\Theta_s)\, F_{\lambda_1\lambda_2\lambda_3\lambda_4}(s,t)\, d^j_{\mu\lambda}(\Theta_s) \ . \qquad /2.9/$$

We do not expect any problem with the analytic continuations of /2.8,9/, if they involve no C.M. amplitudes, but rather ones defined in a frame, in which neither the time-component, or the magnitude of the three-vector part of the total four-momentum, P_μ, are vanishing. The symbol $d^j_{\lambda\mu}$ denotes the familiar Wigner d-functions.

3. COMPLEX ANGULAR MOMENTUM

In this section we are going to describe complex angular momentum theory for unequal-mass scattering, which, on one hand, is related to that for equal-mass scattering as strongly as possible, but, on the other hand, makes use of the remarks of the previous section. Namely, that, first, the scattering amplitude is to be expanded in terms of Poincaré representations in a frame, in which the total four-momentum $P_\mu = p_{1\mu} + p_{2\mu}$ is of the form $/p_o,0,0,p/$. Second, the appropriate variable to be used in a complex angular momentum theory is w rather than j. /Of course, this distinction is irrelevant, when $s \neq 0$, and we shall use the variable j until we do not want to go to $s = 0$./

The crucial points of conventional complex angular momentum theory /see refs. [12, 13]/ are the following:

1. Using Carlson's theorem, one defines two functions over the complex j-plane from the s-channel partial wave amplitudes.

2. By Watson-Sommerfeld transformation one casts the partial wave series into an integral along a curve of the j-plane from $-\frac{1}{2}-i\infty$ to $-\frac{1}{2}+i\infty$.

3. After analytic continuation in the s and t Mandelstam variables one obtains the crossed channel scattering amplitude represented by the background integral /along the line $\mathrm{Re}j = -\frac{1}{2}$/, and the residues of poles appearing on the half-plane $\mathrm{Re}j > -\frac{1}{2}$. /Cuts will not be considered in this paper./ It is assumed, that the contribution of the integral along the infinite half-circle is still negligible.

Now we consider the elastic scattering of two spinless particles with masses m and μ, $p_1^2 = p_3^2 = m^2$, $p_2^2 = p_4^2 = \mu^2$. The Mandelstam variables are:

$$\begin{aligned} s &= (p_1 + p_2)^2 = (p_3 + p_4)^2 , \\ t &= (p_1 - p_3)^2 = (p_2 - p_4)^2 , \qquad /3.1/ \\ u &= (p_1 - p_4)^2 = (p_2 - p_3)^2 , \end{aligned}$$

$$\cos\Theta_s = 1 + \frac{2st}{\Delta(s,m^2,\mu^2)} \qquad /3.2/$$

where Θ_s is the s-channel scattering angle in the C.M. frame, and $\Delta(s,m^2,\mu^2) = [s-(m+\mu)^2][s-(m-\mu)^2]$. In the s-channel the partial wave series for the scattering amplitude $F(s,t)$ looks like

$$F(s,t) = \frac{m\mu s}{(m^2-\mu^2+\Delta^{1/2})^2 - s^2} \sum_{j=0}^{\infty} F(s,j)(2j+1)\, P_j(\cos\Theta_s) \ , \qquad /3.3/$$

where the partial wave amplitudes $F(s,j)$ are defined as follows:

$$F(s,j) = \frac{(m^2-\mu^2+\Delta^{1/2})^2 - s^2}{2m\mu s} \int_{-1}^{1} d(\cos\Theta_s)\, F(s,t)\, P_j(\cos\Theta_s) \ . \qquad /3.4/$$

In the spinless case the d-functions of /2.8/ and /2.9/ are the familiar Legendre-polinomials, $P_j(z)$. The kinematical factor $\frac{sm\mu}{(m^2-\mu^2+\Delta^{1/2})^2 - s^2}$ in /3.3,4/ corresponds to the one $\frac{s}{(|p_o|+p)^2}$ of /2.8,9/. It could have been included into $F(s,j)$, but it has significance when we go to $s = 0$, therefore we prefer to write it explicitly. In the equal-mass case it is only a numerical factor $\frac{1}{4}$.

We assume $F(s,t)$ to satisfy unsubtracted dispersion relation in the variable t at fixed s:

$$F(s,t) \equiv F_t(s,t) + F_u(s,t) =$$

$$= \frac{1}{\pi}\int_{4m^2}^{\infty} dt' \,\frac{A_t(s,t')}{t'-t} + \frac{1}{\pi}\int_{-\infty}^{(m-\mu)^2-s} dt' \,\frac{A_u(s,t')}{t'-t} \ . \qquad /3.5/$$

Correspondingly, we define $F(s,j) \equiv F_t(s,j) + F_u(s,j)$ and obtain:

$$F_t(s,j) = \frac{2}{\pi}\,\frac{(m^2-\mu^2+\Delta^{1/2})^2 - s^2}{m\mu\Delta} \int_{4m^2}^{\infty} dt'\, A_t(s,t')\, Q_j\left(1 + \frac{2st'}{\Delta}\right) \ , \qquad /3.6/$$

$$F_u(s,j) = \frac{2}{\pi}\,\frac{(m^2-\mu^2+\Delta^{1/2})^2 - s^2}{m\mu\Delta} \int_{-\infty}^{(m-\mu)^2-s} dt'\, A_u(s,t')\, Q_j\left(1 + \frac{2st'}{\Delta} - i0\right).$$

/3.7/

In /3.7/ it is also denoted, that the real, less than -1 argument of the Q_j function is to be understood as limit from the lower half-plane.

It is usual at this stage to introduce complex angular momentum. As is well-known, the mathematical problem of defining an analytic function having prescribed values at non-negative integer values of j involves an essential non-uniqueness. The tradition in Regge-pole theory is to look for analytic continuations satisfying the conditions of Carlson's theorem, and this leads to the signatured functions:

$$F_{\pm}(s,j) = F_t(s,j) \mp F_u(s,j) \exp i\pi j \qquad /3.8/$$

We are not going to follow this tradition, but rather we define complex angular momentum directly through /3.6,7/. Some problems arising from the use of $F_{t,u}(s,j)$ instead of $F_{\pm}(s,j)$ will be discussed at the end of this section. The merits of our choice will be clear only from the subsequent ones.

Now, still in the s-channel, we can write integral

$$F_{t,u}(s,t) = \frac{1}{2} \frac{sm\mu}{\left(m^2-\mu^2+\Delta^{1/2}\right)^2 - s^2} \times$$

$$\times \int_C dj\, \frac{(2j+1)}{\sin\pi j} F_{t,u}(s,j)\, P_j\left(-1 - \frac{2st}{\Delta}\right) \qquad /3.9/$$

on the j-plane instead of the original partial wave series. The contour C encircles the positive real half-axis. Until we are in the s-channel all the poles of the integrand in /3.9/ are due to the zeros of $\sin\pi j$ at integer values of j. After analytic continuation into the t or u-channel also the functions $F_{t,u}(s,j)$ have poles at real $j = \alpha(s)$ values. Then also the contribution of these poles is to be included in the expression /3.9/. The basic assumption of Regge-pole phenomenology is that the contribution of these latter poles dominates over the remainder, the contribution of the poles due to $\sin\pi j$. The usual "proof" for this is to deform the contour C into a straight line along $\mathrm{Re}\, j = -\frac{1}{2}$ and an infinite half-circle on the right half-plane. If one assumes that the integral along this half-circle is zero, it is easy to see from the asymptotic expressions for the $P_j(z)$ functions, that for large values of $\cos\Theta_s$ the background integral is reasonably neglected in comparison with the Regge pole contributions. Obviously, this "proof" relies very strongly on the appropriate asymptotic behaviour of the $F(s,j)$ functions in the variable j. In the t or u-channel this cannot be justified simply by looking at the integrated of

/3.6,7/and it cannot be done either for the signatured functions $F_{\pm}(s,j)$ even in the most familiar equal-mass case [12, 13]. On the other hand, the successes of Regge-pole phenomenology serve with a justification of the assumption. In our treatment $F_t(s,j)$ and $F_u(s,j)$ must be well-behaved, instead of $F_{\pm}(s,j)$. This assumption may very well be compatible with phenomenology, since, although there is an $\exp(i\pi j)$ factor present in /3.8/, $F_{t,u}(s,j)$ and $F_{\pm}(s,j)$ may have the suitable properties even simultaneously.

There is only one thing we certainly lost when using $F_{t,u}(s,j)$ instead of $F_{\pm}(s,j)$. Namely, in the case of signatured functions the analogue expressions of /3.6,7/ make possible to prove, that in the s-channel the functions $F_{\pm}(s,j)$ decrease /since the functions $Q_j(z)$ do so/ fast enough so as the contribution of the infinite half-circle be zero. This is not the case with $F_u(s,j)$. However, one must notice, that in the s-channel this problem has no particular significance. The contour integral has no advantages over the partial wave series either one must keep the contribution of the half-circle or need not.

Our final formulas for $F_{t,u}(s,t)$ are:

$$F_{t,u}(s,t) = F^b_{t,u}(s,t) + F^p_{t,u}(s,t) \;, \qquad /3.10/$$

where $F^b_{t,u}(s,t)$ is the background term:

$$F^b_{t,u}(s,t) = \frac{1}{2i} \frac{sm\mu}{\left(m^2-\mu^2+\Delta^{1/2}\right)^2 - s^2} \times$$

$$\times \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} dj \, \frac{(2j+1)}{\sin\pi j} F_{t,u}(s,j) \, P_j\left(-1 - \frac{2st}{\Delta}\right), \qquad /3.11/$$

and $F^p_{t,u}(s,t)$ denotes the Regge-pole part:

$$F^p_{t,u}(s,t) = \frac{1}{2i} \frac{sm\mu}{\left(m^2-\mu^2+\Delta^{1/2}\right)^2 - s^2}$$

$$\times \sum_{\text{poles}} \oint_{C_i} dj \, \frac{(2j+1)}{\sin\pi j} F_{t,u}(s,j) \, P_j\left(-1 - \frac{2st}{\Delta}\right) . \qquad /3.12/$$

As was discussed, we assume the representation /3.10-12/ of the functions $F_{t,u}(s,t)$ to be valid in the t and u-channels.

4. COMPLEX ANGULAR MOMENTUM IN THE $s = 0$ LIMIT

After having fixed our definitions for a complex angular momentum theory at $s \neq 0$, we investigate its limit to $s = 0$, which is a physical point for the u-channel. We make use of the fact, that there is a finite piece of the u-channel physical region above $s = 0$, and in the present paper we restrict ourselves to reaching the point $s = 0$ through positive values of s. That is, we consider the formulas /3.6,7,11,12/ for $s-i0$, $u+i0$, $0 < s < (m-\mu)^2$, $(m+\mu)^2 < u < \frac{(m^2-\mu^2)^2}{s}$, and, keeping u fixed, we let s go to zero. It is worth remarking, that still we are on the lower edge of the cut of the Q_j function in /3.7/.

In the usual treatments the limit $s = 0$ is taken at fixed values of j, and the singularity problem arises due to the singularity of $P_j(z)$ at $z = -1$ and of $Q_j(z)$ at $z = 1$. In our approach w is the fundamental variable, and we calculate the limit keeping w fixed. Indeed, first we introduce a /dimensionless/ variable ε instead of j, which is not singularly connected with w even at $s = 0$:

$$w^2 = sj(j+1) + \tfrac{1}{4}s = \frac{(m^2-\mu^2)^2}{4m\mu}\,\varepsilon^2 \quad . \qquad /4.1/$$

The most economical way to calculate the limit of the Legendre-functions is to use the following integral representations [14]:

$$P_j(z) = \frac{1}{\pi}\int_0^{\pi} \left(z + (z^2-1)^{1/2}\cos\varphi\right)^j d\varphi \;, \qquad |\arg z| < \frac{\pi}{2} \;, \qquad /4.2/$$

$$Q_j(z) = \int_0^{\infty} \left(z + (z^2-1)^{1/2}\cosh t\right)^{-j-1} dt, \qquad |\arg(z\pm 1)| < \pi \quad . \qquad /4.3/$$

At the end of the calculations one recognizes Bessel-functions of the first and third kind in the following forms [15]:

$$J_0(z) = \frac{1}{\pi}\int_0^{\pi} \exp\left(iz\cos\phi\right)d\phi \;, \qquad /4.4/$$

$$K_0(z) = \int_0^{\infty} \exp\left(-z\cosh t\right)dt, \qquad |\arg z| < \frac{\pi}{2} \quad . \qquad /4.5/$$

Also the relations between Hankel's functions and the K-function are useful:

$$H_j^{(1)}(z) = J_j(z) + iY_j(z) = -\frac{2i}{\pi}\exp\left(-ij\frac{\pi}{2}\right)K_j\left(z\exp\left(-i\frac{\pi}{2}\right)\right) \quad , \qquad /4.6/$$

$$H_j^{(2)}(z) = J_j(z) - iY_j(z) = i\,\frac{2}{\pi}\exp\left(ij\frac{\pi}{2}\right)K_j\left(z\exp\left(i\frac{\pi}{2}\right)\right) \quad . \qquad /4.7/$$

Here $Y_j(z)$ stands for the Bessel-functions of the second kind.

First we deal with the limits of the functions $F_{t,u}(s,j)$:

$$\lim_{s\to o} F_t(s,j) \equiv F_t(o,\varepsilon) = \frac{8}{\pi}\frac{1}{m\mu}\int_{4m^2}^{\infty} dt'\, A_t(o,t')\, K_o\left(\varepsilon\sqrt{\frac{t'}{m\mu}}\right) \quad , \qquad /4.8/$$

$$\lim_{s\to o} F_u(s,j) \equiv F_u(o,\varepsilon) = -\frac{4i}{m\mu}\int_{-\infty}^{o} dt'\, A_u(o,t')\, H_o^{(2)}\left(\varepsilon\sqrt{\frac{-t'}{m\mu}}\right) +$$

$$+\frac{8}{\pi}\frac{1}{m\mu}\int_{o}^{(m-\mu)^2} dt'\, A_u(o,t')\, K_o\left(\varepsilon\sqrt{\frac{t'}{m\mu}}\right) \quad . \qquad /4.9/$$

Next we calculate the limit of the background integral term /3.11/. It is easy to see, that

$$\lim_{s\to o}\frac{1}{\sin\pi j}P_j\left(-1-\frac{2st}{\Delta}\right) = \begin{cases} iH_o^{(1)}\left(\varepsilon\sqrt{\frac{-t}{m\mu}}\right) & \text{for} \quad \mathrm{Im}\varepsilon > o \; , \\ -iH_o^{(2)}\left(\varepsilon\sqrt{\frac{-t}{m\mu}}\right) & \text{for} \quad \mathrm{Im}\varepsilon < o \; . \end{cases} \qquad /4.10/$$

This yields:

$$F_{t,u}^{b}(o,t) = \frac{1}{8\pi}\int_{o}^{i\infty} F_{t,u}(o,\varepsilon)\, H_o^{(1)}\left(\varepsilon\sqrt{\frac{-t}{m\mu}}\right)\varepsilon\, d\varepsilon +$$

$$-\frac{1}{8\pi}\int_{-i\infty}^{o} F_{t,u}(o,\varepsilon)\, H_o^{(2)}\left(\varepsilon\sqrt{\frac{-t}{m\mu}}\right)\varepsilon\, d\varepsilon \quad . \qquad /4.11/$$

If $F_{t,u}(0,\varepsilon)$ behave at most like polinomials on the right ε-plane for $|\varepsilon| \to \infty$ eq. /4.11/ can be written also as

$$F^{b}_{t,u}(0,t) = \frac{1}{4\pi}\int\limits_{0}^{\infty} \varepsilon\, d\varepsilon\, F_{t,u}(0,\varepsilon)\, J_{0}\left(\varepsilon\sqrt{\frac{-t}{m\mu}}\right) \qquad /4.12/$$

This last expression looks exactly like an expansion with respect to light-like, unitary representations of the Poincaré group. /Similar result was obtained also in [6]./ Our assumption about the asymptotic behaviour of the function $F_t(0,\varepsilon)$ is obviously correct. The situation is more complicated in the case of $F_u(0,\varepsilon)$. The integral representation /4.9/ defines it only for $\mathrm{Im}\varepsilon < 0$, where our assumption about its asymptotic behaviour can be again verified. For $\mathrm{Im}\varepsilon > 0$ it remains unverified, just like when $s\neq 0$. It will be later shown, however, that the assumptions we made are reasonable.

The calculation of the pole terms leads to an interesting result, if one supposes that at $s=0$ the poles are placed at real $\varepsilon_i(s=0) = \varepsilon_i$ points. Due to /4.10/, the contour integrals of /3.12/ must be evaluated not by the theorem of residues, but by applying the formula:

$$\frac{1}{x \pm i0} = \frac{P}{x} \mp i\pi\delta(x)$$

The result is:

$$F^{p}_{t,u}(0,t) = \sum_{\text{poles}} \beta_{t,u}(\varepsilon_i)\, Y_{0}\left(\varepsilon_i\sqrt{\frac{-t}{m\mu}}\right) \qquad /4.13/$$

where $\beta_{t,u}(\varepsilon_i)$ denotes the residues of the functions $F_{t,u}(0,\varepsilon)$. It is remarkable, that the second kind function Y_0 has appeared in /4.13/.

All the calculations of this section were performed by changing the order of integrations and limiting in s. Obviously, had we not used the functions $F_{t,u}(s,j)$ instead of $F_{\pm}(s,j)$, we should have obtained meaningless results. On the other hand, the limit of the $F_{\pm}(s,j)$ functions may very well exist, even if the limit of the integrands does not. /We remind the reader to theorems, e.g., about the existence of $\lim\limits_{\mu\to\infty}\int\limits_{a}^{b} f(x)\sin\mu x\,dx$./ However, making simply the assumption that $\lim F_{\pm}(s,j)$ exist, the formulas would get uncontrollable.

5. SELF-CONSISTENCY AND COMPATIBILITY WITH DISPERSION RELATIONS

This section is devoted to the examination of two problems. The first is related with the connection of complex angular momentum theory and expansions with respect to Poincaré representations. Our concept /in fact, it is due to Hermann [7]/ is, that complex angular momentum is important even if the scattering amplitude is square integrable: it is a tool to continue into each other the Poincaré expansions of the scattering amplitude for total four-momenta of different character. This interpretation makes use of the fact, that those unitary representations of the Poincaré group, which appear to be relevant for the expansion of square-integrable functions in the time-like, light-like and space-like cases, can be characterized by the eigenvalues of one and the same Casimir operator $W_\mu W^\mu$ /beyond, of course, $P_\mu^2 = s$/. It is not priori obvious, that there exists an analytic function $F(s,w)$, which at the appropriate values of s and w takes the values of the expansion coefficients for the previous three expansions. /It is very difficult to say anything about the effect of non-square-integrability, beyond that they presumably correspond to certain w singularities of $F(s,w)$./

The second question is independent on group-theory, and is probably more important from the point of view of theories based on the well-established analytic properties of the scattering amplitude. Namely, the question arises, whether our prescription for the $s = 0$ limit is compatible with dispersion relations we assumed to be valid also for the u-channel amplitude.

To answer the first question we compare the formulas for the u-channel obtained by the analytic continuation of the Watson-Sommerfeld transformed form of the s-channel scattering amplitude /that is, the formulas /3.6,7,10,11/ and /4.8,9,11,12/, and the appropriate crossed-channel expansions we are going to write down assuming square integrable /in $\cos\Theta_s$!/ scattering amplitude also in the u-channel. Clearly, the main task is to cast the inverse formulas for these latter expansions,

$$F_{t,u}(s,j) = \frac{(m^2-\mu^2+\Delta^{1/2})^2 - s^2}{2sm\mu} \int_{-1}^{1} dc\, F_{t,u}(s,t)\, P_j(c) \quad , \qquad /5.1/$$

and

$$F_{t,u}(0,\varepsilon) = \int_{0}^{\infty} \xi\, d\xi\, F_{t,u}(0,t)\, J_0(\varepsilon\xi) \qquad /5.2/$$

into form comparable with the previously obtained ones of Sects. 2. and 4.

Our notations are:

$$c \equiv \cos\Theta_s = 1 + \frac{2st}{\Delta\left(s,m^2,\mu^2\right)} \quad , \qquad /5.3/$$

$$\xi = \sqrt{\frac{-t}{m\mu}} \qquad /5.4/$$

For this purpose, at $s \neq 0$, we should apply the identity:

$$Q_j(z) = \frac{1}{2}\int_{-1}^{1} dc\, \frac{P_j(c)}{z-c} = \frac{1}{2}\int_{-\frac{\Delta}{\varepsilon}}^{0} \frac{P_j\left(1+\frac{2st}{\Delta}\right)}{t'-t}\, dt, \qquad /5.5/$$

where

$$z = 1 + \frac{2st'}{\Delta\left(s,m^2,\mu^2\right)} \quad . \qquad /5.6/$$

There was no problem with /5.5/ in the s-channel, where we needed it only for $t' - t \neq 0$, $-1 < c < 1$, $|z| > 1$. When we are in the u-channel, in the region $0 < s < (m-\mu)^2$, $(m+\mu)^2 < u < \frac{(m^2-\mu^2)^2}{s}$, the situation changes, and can be summarized as follows. From a detailed study of the criginal Cauchy-integral one can see, that the dispersion relation /3.5/ is to be rewritten as

$$F(s,t) = \frac{1}{\pi}\int_{-\infty}^{\infty} dt'\, \frac{A(s,t')}{t'-t+i0} \quad , \qquad /5.7/$$

where

$$A(s,t') = \begin{cases} A_t(s,t') \quad , & \text{if} \quad t' > 4m^2 \quad , \\ 0, & \text{if} \quad (m-\mu)^2 - s < t' < 4m^2 \quad , \\ A_u(s,t') \quad , & \text{if} \quad t' < (m-\mu)^2 - s \quad , \end{cases} \qquad /5.8/$$

and $\frac{1}{t'-t+i0}$ denotes the generalized function $\frac{P}{t'-t} - i\pi\delta(t'-t)$ [16]. The plus sign of $i0$ in /5.8/ comes from the $i\varepsilon$ -prescription of S-matrix theory. The condition $-1 < c < 1$ remains true, but it is easy to see, that now we need /5.5/ also for values of z on the real axis between -1 and $+1$, when /5.5/ fails to be valid in the sense of classical functions. It remains, however, true in the sense of generalized functions. Namely, it is shown in Appendix A that

$$Q_j(x \pm i0) = \frac{1}{2} \int_{-1}^{1} dc \, \frac{P_j(c)}{x-c\pm i0} \qquad /5.9/$$

is true for $-1 < c < 1$, and for any value of x. Then it is obvious, that the formulas /3.6,7/ appear for the expansion coefficients also in the u-channel. This shows, that starting either from the s or the u-channel, one can define one and the same complex angular momentum. It is clear, moreover, that no simple trick /like the introducing of signatured functions in the s-channel/ makes possible to define analytic continuation satisfying Carlson's theorem. In fact, complex angular momentum functions satisfying Carlson's theorem in the u-channel would be incompatible with the ones defined in the s-channel.

In case of $s = 0$ the basic formula one must apply is [15]:

$$\int_{-\infty}^{0} dt \, \frac{J_0\left(\varepsilon\sqrt{\frac{-t}{m\mu}}\right)}{t' - t} = K_0\left(\varepsilon\sqrt{\frac{t'}{m\mu}}\right) \qquad /5.10/$$

which is valid in classical sense for $|\arg t'| < \pi$. However, it is shown in Appendix A, that for $\arg t' = \pm\pi$ equation /5.12/ remains true, and it is to be understood as

$$\int_{-\infty}^{0} dt \, \frac{J_0\left(\varepsilon\sqrt{\frac{-t}{m\mu}}\right)}{t'-t\pm i0} = \mp i \frac{\pi}{2} H^{\binom{2}{1}}\left(\varepsilon\sqrt{\frac{-t'}{m\mu}}\right) . \qquad /5.11/$$

These relations assure, that the formulas of the crossed channel expansion and the ones obtained in Sect.4. from the s-channel expansion coincide also at $s = 0$.

The problem of compatibility with dispersion relations, mentioned at the beginning of this section can be formulated in the following manner. The expansion procedure followed in the previous sections consists, first, in giving the kernel of /5.7/ the form

$$\frac{1}{t'-t+i0} = \frac{2s}{\Delta} \sum_{j=0}^{\infty} (2j+1) P_j(c) \, Q_j(x+i0) = \frac{s}{i\Delta} \int_C dj \, \frac{(2j+1)}{\sin\pi j} P_j(-c) \, Q_j(x+i0), \qquad /5.12/$$

or, for $s = 0$:

$$\frac{1}{t'-t+i0} = \frac{1}{m\mu}\int_0^\infty \varepsilon\, d\varepsilon\, J_0\left(\varepsilon\sqrt{\frac{-t}{m\mu}}\right) K_0\left(\varepsilon\sqrt{\frac{t'+i0}{m\mu}}\right) \qquad /5.13/$$

Second, substituting the right-hand side of /5.12/ or /5.13/ into /5.7/, and changing the order of integrations, one gets:

$$F_t(s,t) = \frac{1}{i\pi}\frac{s}{\Delta}\int_C dj\,(2j+1)\,\frac{P_j(-c)}{\sin\pi j}\left[\int_{4m^2}^\infty A_t(s,t')\;Q_j\left(1+\frac{2st'}{\Delta}\right)dt'\right] \qquad /5.14/$$

or, when $s = 0$

$$F_t(0,t) = \frac{1}{\pi}\frac{1}{m\mu}\int_0^\infty \varepsilon\, d\varepsilon\, J_0\left(\varepsilon\sqrt{\frac{-t}{m\mu}}\right)\left[\int_{4m^2}^\infty A_t(0,t')\;K_0\left(\varepsilon\sqrt{\frac{t'}{m\mu}}\right)dt'\right] \qquad /5.15/$$

/The reader can easily write down the corresponding expressions for $F_u(s,t)$/. We consider a limiting prescription compatible with the dispersion relation, if the limit of the expression for $\frac{1}{t'-t+i0}$ given at $s > 0$ is identical with the expression given at $s = 0$. Obviously, our prescription has this property.

Another problem arising is whether the changing the order of integrations is a legal step. In fact, this is the question about the convergence of our expansions. We are not going to discuss this delicate problem for $s \neq 0$, where we have the more or less familiar, old formulas. For $s = 0$ and the function $F_t(0,t)$ we state the following theorem: if the function $F_t(0,t)$ can be represented for $t < 0$ by an integral

$$F_t(0,t) = \frac{1}{\pi}\int_{4m^2}^\infty dt'\;\frac{A_t(0,t')}{t'-t}$$

where the discontinuity $A_t(0,t')$ is integrable in any finite interval of $/4m^2,\infty/$, and behaves like t'^{α} for $t'\to\infty$, then the equality /5.15/ is true. /Obviously, this theorem makes possible to write Fourier-Bessel integral for a non-square-integrable class of functions./ The proof of this theorem is given in Appendix B. To get a corresponding theorem also for $F_u(0,t)$ we need further work.

6. DISCUSSION

In the previous sections we described the basic ideas for calculating the limit of the Watson-Sommerfeld transformed scattering amplitude to $s = 0$ in the unequal-mass case. We discuss here some characteristics of our result:

$$F(s=0,t) = \frac{1}{4\pi}\int_0^\infty \varepsilon\, d\varepsilon \left[F_t(0,\varepsilon) + F_u(0,\varepsilon)\right] J_0\left(\varepsilon\sqrt{\frac{-t}{m\mu}}\right) +$$

$$+ \sum_{\text{poles}} \beta(\varepsilon_i)\, Y_0\left(\varepsilon_i\sqrt{\frac{-t}{m\mu}}\right) , \qquad /6.1/$$

where

$$F_t(0,\varepsilon) = \frac{8}{\pi}\frac{1}{m\mu}\int_{4m^2}^\infty dt'\, A_t(0,t')\, K_0\left(\varepsilon\sqrt{\frac{t'}{m\mu}}\right) , \qquad /6.2/$$

and

$$F_u(0,\varepsilon) = -\frac{4i}{m\mu}\int_{-\infty}^0 dt'\, A_u(0,t')\, H_0^{(2)}\left(\varepsilon\sqrt{\frac{-t'}{m\mu}}\right) +$$

$$+ \frac{8}{\pi}\frac{1}{m\mu}\int_0^{(m-\mu)^2} dt'\, A_u(0,t')\, K_0\left(\varepsilon\sqrt{\frac{t'}{m\mu}}\right) . \qquad /6.3/$$

Our first observation is that the pole terms do not exhibit the t^α power behaviour for $(-t)\to\infty$, since the Y_0 functions behave like

$$Y_0(z)\underset{|z|\to\infty}{\sim}\sqrt{\frac{2}{\pi z}}\sin\left(z - \frac{\pi}{4}\right) . \qquad /6.4/$$

On the other hand, the theorem stated at the end of the previous section indicates, that the first, "background" term of /6.1/ is probably sufficient to expand a Regge-behaved scattering amplitude. /In fact, we proved it only for $F_t(0,t)$, but similar statements seem to be valid also for $F_u(0,t)$./ It follows, that if we believe in the t^α asymptotic behaviour, the usual rule concerning the dominance of pole terms over the "background" integral does not apply at $s = 0$. The formulas /6.2,3/ indicate, that the pole terms of /6.1/ are probably not present at all. It is easy to see, that the usual assumptions $A_t(0,t') \underset{t'\to\infty}{\sim} t'^{\alpha}$, $A_u(0,t') \underset{-t'\to\infty}{\sim} (-t')^{\alpha}$, $\alpha < 0$, do not lead to any singularity on the half-plane $\mathrm{Re}\varepsilon > 0$.

To see some details we assume a very simple model:

$$A_t(0,t') = \begin{cases} a_t t'^{\alpha}, & \text{if} \quad t' > 0, \\ 0, & \text{if} \quad t' < 0, \end{cases} \qquad /6.5/$$

$$A_u(0,t') = \begin{cases} 0, & \text{if} \quad t' > 0, \\ a_u(-t')^{\alpha}, & \text{if} \quad t' < 0, \end{cases} \qquad /6.6/$$

where $-\frac{1}{2} < \alpha < 0$, a_t and a_u are real constants. The integrals corresponding to /6.2/ and /6.3/ yield:

$$F_t(0,\varepsilon) = \frac{8}{\pi}\frac{a_t}{m\mu}\int_0^{\infty} dt'\ t'^{\alpha}\ K_0\left(\varepsilon\sqrt{\frac{t'}{m\mu}}\right) = \frac{16a_t}{\pi}\,(4m\mu)^{\alpha}\ \Gamma^2(\alpha+1)\ \varepsilon^{-2(\alpha+1)}\ , \qquad /6.7/$$

$$F_u(0,\varepsilon) = -\frac{4ia_u}{m\mu}\int_{-\infty}^{0} dt'(-t')^{\alpha}\ H_0^{(2)}\left(\varepsilon\sqrt{\frac{-t'}{m\mu}}\right) = -\frac{16a_u}{\pi}\ e^{-i\pi\alpha}(4m\mu)^{\alpha}\ \Gamma^2(\alpha+1)\ \varepsilon^{-2(\alpha+1)}.$$

We remind the reader, that the integral /6.3/ defines $F_u(0,\varepsilon)$ only for $\mathrm{Im}\varepsilon < 0$. After evaluating the integral in this region the result can be extended also for $\mathrm{Im}\varepsilon > 0$. /Eq. /6.8/ is just an example for this./ Finally, the integral gives the expected result for the scattering amplitude:

$$F(0,t) = \frac{1}{\sin\pi\alpha}\left[a_t(-t)^{\alpha} - a_u t^{\alpha}\right]. \qquad /6.9/$$

One could examine more complicated models /with more complicated A_t, A_u functions, but with the previous asymptotic behaviour/, but the following features of this simple model would remain unchanged. There are no poles of the functions $F_{t,u}(0,\varepsilon)$ on the right ε half-plane. Instead, one always finds a branch-point at $\varepsilon = 0$ with the characteristic power $\varepsilon^{-2(\alpha+1)}$. For large values of $|t|$ the dominant contribution to the integral /6.1/ comes from the lower end of the integration path, and asymptotically, the form /6.9/ is always reproduced.

It is worth noticing, how nicely these results correspond to the Lorentz pole picture, the usual solution of the singularity problem at $s=0$. First, we have seen, that the "cause" of the t^{α} behaviour is "concentrated" to the $\varepsilon = 0$ point, which is the image of the j-plane /due to the singular mapping at $s=0$/. That is, the power behaviour is something deeply connected with the j-plane. Second, it is not very hard to imagine, that the infinite sequence of the /j-plane/ daughters accumulates /on the ε-plane/ when $s=0$, and forms a branch-point at $\varepsilon=0$. Of course, it is difficult to guess the nature of the branch-point. Just like in case of considering all the conspiring daughters, we did not find here any singularity at $s=0$, only the t^{α} behaviour was reproduced.

Our last remark concerns signature. In the previous sections it was important, that we did not introduce signatured functions. Of course, eq. /3.8/ always makes possible to restore the old formalism with signature if $s\neq 0$. For $s=0$ eq. /3.8/ gets singular. However, introducing the quantities

$$a_{\pm} = \frac{1}{2}\left(a_t \mp a_u\right) \quad , \qquad /6.10/$$

eq. /6.9/ can be rewritten as follows:

$$F(0,t) = \frac{1+e^{-i\pi\alpha}}{\sin\pi\alpha}\, a_{+}(-t)^{\alpha} + \frac{1-e^{-i\pi\alpha}}{\sin\pi\alpha}\, a_{-}(-t)^{\alpha} \quad . \qquad /6.11/$$

Of course, this form follows for $F(0,t)$ from the assumptions of power behaviour and symmetry between the t and u-channels. More remarkable is that our formalism is compatible with it without superimposing to the formulas /notice the factor $\exp(-i\pi\alpha)$ in /6.8//.

ACKNOWLEDGEMENTS

The authors are pleased to thank Professors G.F.Chew, M.Jacob, A.Martin and Ya.A.Smorodinsky for useful discussions and suggestions, which helped them very much in clarifying several problems.

APPENDIX A

In Section 5. we stated the equalities

$$Q_j(x \pm i0) = \frac{1}{2} \int_{-1}^{1} dc \, \frac{P_j(c)}{x-c\pm i0} \quad , \qquad /A.1/$$

and

$$\sum_{j=0}^{\infty} (2j+1)\, P_j(c)\, Q_j(x\pm i0) = \frac{1}{x-c\pm i0} \quad . \qquad /A.2/$$

for $-1 < c < +1$. Their proof is straightforward by using the identity

$$\frac{1}{x-c\pm i0} = \frac{P}{x-c} \mp i\pi\delta(x-c) \quad ,$$

and the formula 15.3/6/ of ref. [17]:

$$P \int_{-1}^{1} dc \, \frac{P_j(c)}{x-c} = \frac{1}{2} Q_j(x)$$

/P denotes the principal value of the integral/, and the one 3.4/9/ of ref. [14]:

$$Q_j(x\pm i0) = Q_j(x) \mp \frac{i\pi}{2} P_j(x) \ .$$

Next we investigate the expression

$$\frac{m\mu}{t'-t\pm i0} = \mp \frac{i\pi}{2} \int_0^{\infty} \varepsilon \, d\varepsilon \, J_0\left(\varepsilon\sqrt{\frac{-t}{m\mu}}\right) H_0^{\binom{2}{1}}\left(\varepsilon\sqrt{\frac{-t'}{m\mu}}\right) \quad , \qquad /A.3/$$

where both t and t' are negative. We apply the regularization technique of ref. [16], and define the integral /A.3/ as follows:

$$\frac{m\mu}{t'-t\pm i0} = \mp \frac{i\pi}{2} \lim_{s\to 2} \int_0^{\infty} \varepsilon^{s-1} \, J_0\left(\varepsilon\sqrt{\frac{-t}{m\mu}}\right) H_0^{\binom{2}{1}}\left(\varepsilon\sqrt{\frac{-t'}{m\mu}}\right) d\varepsilon \quad . \qquad /A.4/$$

It was obtained in ref. [11]:

$$\lim_{s\to 2} \int_0^\infty d\varepsilon\ \varepsilon^{s-1}\ J_o\left(\varepsilon\sqrt{\frac{-t}{m\mu}}\right) J_o\left(\varepsilon\sqrt{\frac{-t'}{m\mu}}\right) = 2m\mu\delta(t-t') \quad . \qquad /A.5/$$

The remaining task is to calculate the quantity:

$$\lim_{s\to 2} \int_0^\infty d\varepsilon\ \varepsilon^{s-1}\ J_o\left(\varepsilon\sqrt{\frac{-t}{m\mu}}\right) Y_o\left(\varepsilon\sqrt{\frac{-t'}{m\mu}}\right) \quad .$$

For this purpose one must use the formulas 6.8/37,38,47/ of ref. [18]:

$$\int_0^\infty x^{s-1}\ J_o(bx)\ Y_o(ax)dx = \frac{1}{\pi}\ 2^{s-1}\ a^{-s}\ \sin\frac{\pi}{2}(s-1)\ \Gamma^2\left(\frac{s}{2}\right) F\left(\frac{s}{2},\ \frac{s}{2};\ 1;\ \frac{b^2}{a^2}\right) \quad ,$$

if $a > b > 0$, $0 < \mathrm{Re}s < 2$.

$$\int_0^\infty x^{s-1}\ J_o(bx)\ Y_o(ax)dx = -\int_0^\infty x^{s-1}\ J_o(ax)\ Y_o(bx)dx -$$

$$- \frac{4}{\pi^2}\cos\frac{\pi}{2}(s-1) \int_0^\infty x^{s-1}\ K_o(bx)\ K_o(ax)dx \quad ,$$

if $b > a > 0$, $0 < \mathrm{Re} < 2$.

$$\int_0^\infty x^{s-1}\ K_o(ax)\ K_o(bx)dx = \frac{2^s\ a^{-s}}{\Gamma(s)}\ \Gamma^4\left(\frac{s}{2}\right) F\left(\frac{s}{2},\ \frac{s}{2};\ s;\ 1 - \frac{b^2}{a^2}\right) \quad ,$$

if $\mathrm{Re}\ a+b > 0$, $\mathrm{Re}\ s > 0$.

The result is:

$$-i\ \frac{\pi}{2} \lim_{s\to 2} \int_0^\infty \varepsilon^{s-1}\ J_o\left(\varepsilon\sqrt{\frac{-t}{m\mu}}\right) Y_o\left(\varepsilon\sqrt{\frac{-t'}{m\mu}}\right) d\varepsilon = m\mu\ \frac{P}{t'-t} \quad , \qquad /A.6/$$

which completes the proof of /A.3/.

APPENDIX B

In this Appendix we are going to prove the theorem stated at the end of Section 5. The theorem is as follows:
If

$$F(t) = \int_{t_o}^{\infty} \frac{A(t')}{t'-t}\, dt' \quad , \qquad /B.1/$$

where $A(t')$ is integrable in any finite interval of $/t_o,\infty/$, and $|A(t')| \underset{t'\to\infty}{\sim} t'^{\alpha}$, then

$$F(t) = \int_{t_o}^{\infty} A(t') \left[\int_0^{\infty} \varepsilon\, J_o(\varepsilon\sqrt{-t})\, K_o(\varepsilon\sqrt{t'})\, d\varepsilon\right] dt' =$$

$$= \int_0^{\infty} J_o(\varepsilon\sqrt{-t})\, \varepsilon \left[\int_{t_o}^{\infty} A(t')\, K_o(\varepsilon\sqrt{t'})\, dt'\right] d\varepsilon \quad . \qquad /B.2/$$

The proof will be performed in two steps. First we prove, that

$$\int_{t_o}^{\infty} A(t') \left[\int_0^{\infty} \varepsilon\, J_o(\varepsilon\sqrt{-t})\, K_o(\varepsilon\sqrt{t'})\, d\varepsilon\right] dt' =$$

$$= \lim_{\Omega\to o} \int_{t_o}^{\infty} A(t') \left[\int_0^{1/\Omega} \varepsilon\, J_o(\varepsilon\sqrt{-t})\, K_o(\varepsilon\sqrt{t'})\, d\varepsilon\right] dt' \quad . \qquad /B.3/$$

Second, we show, that

$$\int_{t_o}^{\infty} A(t') \left[\int_0^{1/\Omega} \varepsilon\, J_o(\varepsilon\sqrt{-t})\, K_o(\varepsilon\sqrt{t'})\, d\varepsilon\right] dt' =$$

$$= \int_0^{1/\Omega} J_o(\varepsilon\sqrt{-t})\, \varepsilon \left[\int_{t_o}^{\infty} A(t')\, K_o(\varepsilon\sqrt{t'})\, dt'\right] d\varepsilon \quad . \qquad /B.4/$$

Combining /B.3/ and /B.4/ we just obtain /B.2/.

The proof is based on two well-known theorems of mathematical analysis:

1/ If the function $f(x,y)$ can be written as $f(x,y) = g(x,y)\,k(y)$, where $k(y)$ is integrable in any finite interval of $\alpha < y < +\infty$, $g(x,y)$ is continuous in $a \leq x \leq b$, $\alpha \leq y < \infty$, and the integral

$$F(x) = \int_{\alpha}^{\infty} f(x,y)dy$$

converges uniformly in $[a,b]$, then the function $F(x)$ is continuous in $[a,b]$, and, under the same assumptions, the following equality is true:

$$\int_a^b \left[\int_{\alpha}^{\infty} f(x,y)dy\right]dx = \int_{\alpha}^{\infty}\left[\int_a^b f(x,y)dx\right]dy \quad .$$

2/ The integral $\int_{\alpha}^{\infty} f(x,y)dy$ converges uniformly in $[a,b]$, if there exists a function $G(y)$ such, that $|f(x,y)| \leq G(y)$, for any $a \leq x \leq b$, $\alpha \leq y$, and the integral $\int_{\alpha}^{\infty} G(y)dy$ converges.

First we define two functions:

$$f_t(t',\Omega) \equiv \int_0^{1/\Omega} x\, J_o(x)\, K_o\left(x\sqrt{-\frac{t'}{t}}\right)dx \quad , \qquad /B.5/$$

and

$$F(t,\Omega) \equiv -\frac{1}{t}\int_{t_o}^{\infty} f_t(t',\Omega)\, A(t')dt' \quad . \qquad /B.6/$$

Obviously,

$$F(t) = F(t,0) \equiv -\frac{1}{t}\int_{t_o}^{\infty} f_t(t',0)\, A(t')dt' \quad . \qquad /B.7/$$

We show, that $F(t,\Omega)$ is continuous function of Ω in some $0 \leq \Omega \leq \Omega_o$ interval, where Ω_o is arbitrary positive number. The function $A(t')$ is, by assumption, integrable in any finite interval of $/t_o,\infty/$. It is easy to see, that the integral in /B.6/ is uniformly convergent in $0 \leq \Omega \leq \Omega_o$. Namely,

$$\left|f_t(t',\Omega)\,A(t')\right| \leq \left|A(t')\right| \int_0^\infty \left|xK_o\left(x\sqrt{\frac{-t'}{t}}\right)\right|dx = -\frac{t}{t'}\left|A(t')\right| \int_0^\infty \left|yK_o(y)\right|dy \quad .$$

The last integral is convergent, thus we have the relation

$$-\frac{1}{2}\left|f_t(t',\Omega)\,A(t')\right| \leq C\,\frac{|A(t')|}{t'}$$

valid for any Ω in $[0,\Omega_o]$, and t' in $[t_o,\infty)$. It is assumed, that $|A(t')| \underset{t'\to\infty}{\sim} t'^{\alpha}$, $\alpha < 0$, therefore also the integral

$$\int_{t_o}^\infty \frac{|A(t')|}{t'}\,dt'$$

converges. Consequently, the integral /B.6/ converges uniformly. Lengthy, but straightforward calculation yields, that the difference

$$|f_t(t'+\delta,\ \Omega+\omega) - f_t(t',\Omega)| \leq$$

$$\leq |f_t(t'+\delta,\ \Omega+\omega) - f_t(t'+\delta,\ \Omega)| + |f_t(t'+\delta,\ \Omega) - f_t(t',\ \Omega)|$$

can be made arbitrarily small for any $0 \leq \Omega \leq \Omega_o$, $t_o \leq t'$. It follows, that $F(t,\Omega)$ is continuous function of Ω in $0 \leq \Omega \leq \Omega_o$, that is, $\lim_{\Omega\to 0} F(t,\Omega) = F(t,0)$. The equality /B.3/ is proved.

Before starting with the proof of /B.4/, we introduce a new auxiliary function:

$$f(t',\Omega,E) = \int_E^{1/\Omega} xJ_o(x)\,K_o\left(x\sqrt{-\frac{t'}{t}}\right)dx \quad , \qquad /B.8/$$

and consider

$$F(t,\Omega,E) = -\frac{1}{t}\int_{t_o}^\infty A(t')\,f(t',\Omega,E)\,dt' \quad . \qquad /B.9/$$

One can prove again, that, at fixed t and Ω, the function $F(t,\Omega,E)$ is continuous function of E in $0 \leq E \leq E_o$, that is,

$$F(t,\Omega,0) \equiv F(t,\Omega) = \left(-\frac{1}{t}\right) \lim_{E\to 0} \int_{t_o}^{\infty} dt'\, A(t') \left[\int_{E}^{1/\Omega} xJ_o(x)\, K_o\left(x\sqrt{-\frac{t'}{t}}\right)dx\right] . \qquad /B.10/$$

Repeating the same reasoning as previously it is easy to check, that

$$\int_{t_o}^{\infty} A(t')\left[\int_{E}^{1/\Omega} xJ_o(x)\, K_o\left(x\sqrt{-\frac{t'}{t}}\right)dx\right]dt' =$$

$$= \int_{E}^{1/\Omega} xJ_o(x) \left[\int_{t_o}^{\infty} A(t')\, K_o\left(x\sqrt{-\frac{t'}{t}}\right)dt'\right]dx \quad . \qquad /B.11/$$

From equations /B.8,9/ and /B.3/ the validity of /B.2/ follows.

61.919

REFERENCES

[1] J.F.Boyce, J.Math.Phys., 8, 675 /1967/.

[2] D.I.Olive, Nucl. Phys., 15B, 617 /1970/.

[3] P.Goddard, A.R.White, Nuovo Cim., 1A, 645 /1971/.

[4] C.Cronström: Generalized /O(2,1) expansions. Talk presented at the Symposium on De Sitter and Conformal Groups, University of Colorado, Boulder, 1970; and

C. Cronström, W.H.Klink, Princeton preprint, February 1971.

[5] F.T.Hadjioannou, Nuovo Cim., 44, 185 /1966/.

[6] G.Feldman, P.Matthews, Phys.Rev. 168, 1587 /1968/.

[7] R.Hermann, "Fourier Analysis on Groups and Partial Wave Analysis", Benjamin, New York, 1970.

[8] K.Szegő, K.Tóth, Nuovo Cim. 66A, 371 /1970/.

[9] K.Szegő, K.Tóth, Annals of Phys., in print, and KFKI-71-38 preprint, Budapest, 1971.

[10] J.F.Boyce et al., IC/67/9 preprint, Trieste, 1967.

[11] K.Szegő, K.Tóth, J.Math.Phys., 12, 846 /1971/.

[12] R.Oehme, Complex angular momentum in elementary particle scattering, in "Strong Interactions and High Energy Physics" /R.G.Moorhouse, Ed./, Oliver and Boyd, London, 1964.

[13] P.D.B.Collins, E.J.Squires, "Regge Poles in Particle Physics", Springer, Berlin, 1968.

[14] H.Bateman, "Higher Transcendental Functions", Vol.1., Mc Graw Hill, New York, 1953.

[15] H.Bateman, "Higher Transcendental Functions", Vol.2., Mc Graw Hill, New York, 1953.

[16] I.M.Gel'fand, G.E.Shilov, "Generalized Functions", Vol.1., Academic, New York, 1964.

[17] H.Bateman, "Tables of Integral Transforms", Vol.1., Mc Graw Hill, New York, 1954.

[18] H.Bateman, "Tables of Integral Transforms", Vol.2., Mc Graw Hill, New York, 1954.

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Készült a KFKI sokszorositó üzemében,
Budapest, 1971. november hó