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REDUCTION OF THE PROBLEM OF REARRANGEMENT PROCESSES TO A PAIR OF TWO-PARTICLE PROBLEMS

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ABSTRACT

The solution of the problem of scattering between complex nuclei can be reduced by Feshbach's method to the solution of an effective two-particle problem. In this paper an extension of Feshbac''s procedure to the case of rearrangement processes is proposed.

It is shown that the problem of rearrangement can be traced back to a pair of scattering processes in the input and the output channel, respectively. The inhomogeneous integral equation for the two-particle radial wave function in the output channel is set up and solved formally by taking into account the boundary conditions represented by the incident wave.

РЕЗЮМЕ

Уже Фешбах показал, как получается амплитуда рассеяния между составными ядрами путем решения эффективной двухчастичной проблемы. В этой работе метод Фешбаха обобщен для процессов перегруппировки. Показано, что проблему перегруппировки можно привести к решению проблем двух эффективных двухчастичных рассеяний, одно из которых происходит в входном, а второе в выходном канале. В работе установлено и формально решено неоднородное интегральное уравнение для выходного канала с краевыми условиями, полученными из начального состояния входного канала.

KIVONAT

Feshbach adott eljárást arra, miként számitható ki összetett magok közötti szórásfolyamat szórási amplitudója egy effektiv kétrészecske-probléma megoldásával. Dolgozatunkban Feshbach módszerét átrendeződéses folyamatokra általánositjuk. Kimutatjuk, hogy az átrendeződés esete visszavezethető két effektiv kétrészecske-probléma megoldására, amelyik közül az egyik a bemenő , a másik a kimenő csatornában zajlik le. Felállitjuk és formálisan megoldjuk a kimenő csatornabeli kétrészecske-állapot inhomogén integrálegyenletét, a bemenő csatornabeli kezdőállapot - megszabta határfeltételekkel.

Feshbach [1] was the first to introduce a two-particle problem that can be considered to be dynamically equivalent to the problem of elastic or inelastic scattering between complex nuclei. The effective two-particle wave function was found to be the projection of the many-particle scattering state onto the open channels of the fragmentation in question; further, it proved to satisfy an integro-differential equation with an energy-dependent non-local effective two-particle potential. To construct the latter quantity, knowledge of the interaction operator and of the complete set of the channel wave functions is required. The reducibility of a many-particle process to a <u>single</u> two-particle problem, however, seems to be restricted to <u>scattering</u> processes. Nevertheless, this does raise the question of whether or not the solution of a <u>rearrangement</u> process may be linked with the solutions of a <u>pair</u> of two-particle processes in the input and output channels, respectively.

It is convenient to take as the starting point for this approach the Rosenfeld-Humblet approximation [2] to the S-matrix element for the transition between channels $a(\alpha)$ and $b(\beta)$ of the fragmentations α and β

$$S_{ba} \approx \frac{k_a}{k_b} \frac{W_{\rho_\beta}\left\{u_{ba}(r_\beta);\, w_b^{(-)}(r_\beta)\right\}}{W_{\rho_\alpha}\left\{u_{aa}(r_\alpha);\, w_a^{(+)}(r_\alpha)\right\}} \qquad (1)$$

Here, u_{aa} and u_{ba} are the projections of the many-particle scattering state ψ_a onto channels a and b, respectively; $w_a^{(+)}$ and $w_b^{(-)}$ denote free or Coulomb-distorted radial wave functions of the outgoing and the incoming type. Approximation (1) may be made asymptotically exact [3] by increasing the channel radii ρ_α and ρ_β to infinity. For $\alpha=\beta$, $a=b$ formula (1) proves at once the equivalence of the many-particle elastic scattering and Feshbach's effective two-particle problem. The function u_{aa} is, in fact, the effective two-particle state for elastic scattering and, in principle, may be obtained by Feshbach's method. To complete the calculation of S_{ba} for the general case, one has still to develop a procedure for obtaining the "mixed" projection u_{ba} for a final fragmentation β that is different from α, the initial one. Below, an argument is proposed for how this may possibly be realized.

The Hamiltonian of the system and the expansion of the scattering state in terms of the channel coordinates and the channel wave functions of the final fragmentation may be written down in the usual notation as

$$H = T_\beta(\underline{r}_\beta) + H_\beta(\xi_\beta) + V_\beta(\underline{r}_\beta, \xi_\beta) \tag{2}$$

and

$$\psi_a^{\,+} = \sum_{b'(\beta)=1}^{N} r_\beta^{-1}\, u_{b'a}(r_\beta)\, \phi_{b'}(\hat{\underline{r}}_\beta, \xi_\beta)$$

$$+ \int_0^\infty d\varepsilon_{b'} \left\{ \sum_{\bar{b}'(\varepsilon_{b'})} r_\beta^{-1}\, u_{\bar{b}'a}(r_\beta, \varepsilon_{b'})\, \phi_{\bar{b}'}(\hat{\underline{r}}_\beta, \xi_\beta, \varepsilon_{b'}) \right\} \tag{3}$$

where N_β is the number of the discrete channels in fragmentation β, and $\bar{b}'(\varepsilon_{b'})$ is the symbol for the set of discrete quantum numbers required to specify the channels for the degenerate continuum eigenvalue ε_b' , which is the sum of the internal excitation energies in channel b'. Inclusion of the continuous channels for any energy is necessitated by the requirement of completness. The channel selection may be simply

$$|b'\rangle \equiv \phi_{b'}(\hat{\underline{r}}_\beta, \xi_\beta) = Y_{\lambda_{b'}}^{\ell_{b'}}(\hat{\underline{r}}_\beta)\, \chi_{b'}(\xi_\beta) \tag{4}$$

where $\chi_{b'}$ stands for the internal state. For the Schrödinger equation of the system we have

$$\left\{T_\beta(\underline{r}_\beta) + H_\beta(\xi_\beta) + V_\beta(\underline{r}_\beta, \xi_\beta) - E\right\}\left\{\sum_{b'} r_\beta^{-1}\, u_{b'a}(r_\beta)\, \phi_{b'}(\hat{\underline{r}}_\beta, \xi_\beta)\right.$$

$$\left. + \int_0^\infty d\varepsilon_{b'} \left[\sum_{\bar{b}'(\varepsilon_{b'})} r_\beta^{-1}\, u_{\bar{b}'a}(r_\beta, \varepsilon_{b'})\, \phi_{\bar{b}'}(\xi_\beta, \varepsilon_{b'})\right]\right\} = 0. \tag{5}$$

This equation must be multiplied from the left with each of the $N_\beta + 1$ components in turn of the symbolic vector

$$\left|\phi_{b''}(\hat{\underline{r}}_\beta, \xi_\beta), \, (b''=b_1, b_2, \ldots, b_{N_\beta});\ \int_0^\infty d\varepsilon_{b'} \sum_{\bar{b}''} \phi_{b''}(\hat{\underline{r}}_\beta, \xi_\beta)\right\rangle \tag{6}$$

and, within each equation of the set thus obtained, integrated over the entire range of the variables $\hat{\underline{r}}_\beta$ and ξ_β. Let us introduce the notation

$$D_{b'}(r_\beta) \equiv \frac{d^2}{dr_\beta^2} - \frac{\ell_{b'}(\ell_{b'}+1)}{r_\beta^2} \tag{7}$$

as well as

$$k_{b'}^2 \equiv \frac{2\mu_\beta}{h^2}\left(E - \varepsilon_{b'}\right) \tag{8}$$

and

$$v_{b''b'}(r_\beta) \equiv \frac{2\mu_\beta}{h^2} \langle \phi_{b''} | V_\beta | \phi_{b'} \rangle$$

$$v_{\bar{b}''b'}(r_\beta, \varepsilon_{b''}) \equiv \frac{2\mu_\beta}{h^2} \langle \phi_{\bar{b}''}(\varepsilon_{b''}) | V_\beta | \phi_{b'} \rangle$$

$$v_{\bar{b}''\bar{b}'}(r_\beta, \varepsilon_{b''}, \varepsilon_{b'}) \equiv \frac{2\mu_\beta}{h^2} \langle \phi_{\bar{b}''}(\varepsilon_{b''}) | V_\beta | \phi_{\bar{b}'}(\varepsilon_{b'}) \rangle \tag{9}$$

The set of coupled equations may now be divided into three groups. First, for the channel $b'' = b$ one has a single equation

$$\left\{ k_b^2 + D_b(r_\beta) - v_{bb}(r_\beta) \right\} u_{ba}(r_\beta) = 0 \tag{10}$$

Second, for the remaining bound channels a set b'' of $N_\beta - 1$ equations is obtained:

$$\sum_{b' \neq b} \left\{ \delta_{b''b'} \left[k_{b'}^2 + D_{b'}(r_\beta) - v_{b''b'}(r_\beta) \right\} u_{b'a}(r_\beta) + \int_0^\infty d\varepsilon_{b'} \left\{ \sum_{\bar{b}'} v_{b''\bar{b}'}(r_\beta, \varepsilon_{b'})\, u_{\bar{b}'a}(r_\beta, \varepsilon_{b'}) \right\} = v_{b''b}(r_\beta)\, u_{ba}(r_\beta) ;$$

$$b''(\beta) \neq b \tag{11}$$

Finally, for the last component of the vector (6) one has the single equation corresponding to the continuum

$$\int_0^\infty d\varepsilon_{b''} \sum_{\bar{b}''} \left\{ \sum_{b'} v_{\bar{b}''b'}(r_\beta,\varepsilon_{b''})\, u_{b'a}(r_\beta) + \right.$$

$$\left. \int_0^\infty d\varepsilon_{b'} \sum_{\bar{b}'} \left[\delta_{\bar{b}''\bar{b}'}\, \delta(\varepsilon_{b''} - \varepsilon_{b'})\left(k^2_{b'} + D_{b'}(r_\beta)\right) + v_{b''b'}(r_\beta,\varepsilon_{b''},\varepsilon_{b'})\right] \omega_{\bar{b}'a}(r_\beta,\varepsilon_{\bar{b}'})\right\}$$

$$= \int_0^\infty d\varepsilon_{b''} \sum_{\bar{b}''} v_{\bar{b}''b}(r_\beta,\varepsilon_{b''})\, u_{ba}(r_\beta) \qquad (12)$$

It is convenient to introduce the vectors $\underline{v}_{.b}$ and $\underline{u}_{.a}$ in the space subtended by the channels $b'(\beta)$, excluding channel b, as

$$(\underline{v}_{.b})_{b'} \equiv v_{b'b}(r_\beta) \qquad \text{for bound channels} \quad b' \neq b$$
$$\equiv v_{\bar{b}'b}(r_\beta,\varepsilon_{b'}) \qquad \text{for the continuum} \qquad (13)$$

and

$$(\underline{u}_{.a})_{b'} \equiv u_{b'a}(r_\beta) \qquad \text{for bound channels} \quad b' \neq b$$
$$\equiv u_{\bar{b}'a}(r_\beta,\varepsilon_{b'}) \qquad \text{for the continuum} \qquad (14)$$

Similarly, in the same truncated space $b',b'' \neq b$, we introduce the matrix $\underline{\underline{v}}$ in the form

$$(\underline{\underline{v}})_{b''b'} \equiv v_{b''b'}(r_\beta) \qquad \text{for bound channels} \quad b',b'' \neq b,$$
$$\equiv v_{\bar{b}''b'}(r_\beta,\varepsilon_{b''}) \qquad \text{for } b' \text{ bound, } b'' \text{ continuous, etc;} \qquad (15)$$

and the diagonal matrices

$$\underline{\underline{D}}\ ,\quad \underline{\underline{k}} \quad \text{and} \quad i\underline{\underline{\varepsilon}} \qquad (16)$$

with obvious diagonal elements.

With eqs. (13) - (16) in mind, eq. (10) may be rewritten as

$$\left\{k^2_b + D_b(r_\beta) - v_{bb}(r_\beta)\right\} u_{ba}(r_\beta) = \underline{v}_{.b}(r_\beta)\cdot\underline{u}_{.a}(r_\beta) \qquad (17)$$

Eqs. (11) -(12) can likewise written down in the compact form

$$\left\{\underline{\underline{k}}^2 + \underline{\underline{D}}(r_\beta) - \underline{\underline{v}}(r_\beta)\right\} \underline{u}_{.a} = \underline{v}_{.b}\, u_{ba}(r_\beta) . \qquad (18)$$

The inhomogeneous vector differential equation (18) solves for $\underline{u}_{.a}$ as

$$\underline{u}_{.a}(r_\beta) = \underline{\omega}_{.a}(r_\beta) + \frac{1}{\underline{\underline{k}}^2 + \underline{\underline{D}}(r_\beta) - \underline{\underline{v}}(r_\beta) + i\underline{\underline{\varepsilon}}}\, \underline{v}_{.a}(r_\beta)\, u_{ba}(r_\beta) \qquad (19)$$

Here the newly introduced ω-s should not in general vanish identically, since, in contrast to the input fragmentation α , all the channels of any other fragmentation involve incident waves. By virtue of eqs. (18) - (19) the ω-s satisfy the vector differential equation

$$\left\{\underline{\underline{k}}^2 + \underline{\underline{D}}(r_\beta) - \underline{\underline{v}}(r_\beta)\right\} \underline{\omega}_{.a}(r_\beta) = 0 \qquad (20)$$

which immediately may be converted into the coupled set of integral equations

$$\underline{\omega}_{.a}(r_\beta) = \underline{n}_{.a} \circ \underline{w}^{(0)}(r_\beta) + \frac{1}{\underline{\underline{k}}^2 + \underline{\underline{D}}(r_\beta) + i\underline{\underline{\varepsilon}}}\, \underline{\underline{v}}(r_\beta)\, \underline{\omega}_{.a}(r_\beta) . \qquad (21)$$

Here, the boundary conditions involved in the first term on the right hand side should yield all the necessary information on the input channel a. The normalization vector $\underline{n}_{.a}$ is meant to carry these data and the vector $\underline{w}^{(0)}$ is regarded as comprising the force-free solutions in channels $b'(\beta) \neq b$ of a definite norm /e.g. in the case of neutral channels $k_b\, r_\beta$ - times the Bessel functions/.

By representing the operator acting on the vector $\underline{\omega}_{.a}$ on the right hand side of eq. (21) by a single matrix, say $\underline{\underline{h}}$, it is easy to see that the dependence of $\underline{\omega}_{.a}$ on the normalization vector $\underline{n}_{.a}$ may be put into the form

$$\underline{\omega}_{.a}(r_\beta) = \underline{\underline{\Omega}}(r_\beta) \cdot \underline{n}_{.a} \qquad (22)$$

where the elements of $\underline{\underline{\Omega}}$ may be constructed in each particular case out of the components of $\underline{w}^{(o)}$ and the elements of $\underline{\underline{h}}$. The essential content of eq. (22) is the way its structure displays how $\underline{\omega}_{.a}$ depends on a. The required information on ϕ_a is thus reduced to the knowledge of the constants $\underline{n}_{.a}$. By inserting relationship (22) into eq. (19), the latter may be combined with eq. (17) to give a scalar integro-differential equation for u_{ba} in terms of the single variable r_β:

$$\left\{k_b^2 + D_b(r_\beta) - v_b^{eff}(r_\beta)\right\} u_{ba}(r_\beta) = \underline{v}^{+}_{.b}(r_\beta)\, \underline{\underline{\Omega}}(r_\beta)\, \underline{n}_{.a} \tag{23}$$

where the concept of the effective two-particle potential was introduced by

$$v_b^{eff}(r_\beta) \equiv v_{bb}(r_\beta) + \underline{v}_{.b}(r_\beta) \frac{1}{\underline{\underline{k}}^2 + \underline{\underline{D}}(r_\beta) - \underline{\underline{v}}(r_\beta) + i\underline{\underline{\varepsilon}}} \underline{v}_{.b}(r_\beta) \quad . \tag{24}$$

The next step is to solve eq. (23) for u_{ba} :

$$u_{ba}(r_\beta) = \tilde{\omega}_{ba}(r_\beta) + \frac{1}{k_b^2 + D_b(r_\beta) - v_b^{eff}(r_\beta) + i\varepsilon} \underline{v}_{.b}(r_\beta)\, \underline{\underline{\Omega}}(r_\beta) \underline{n}_{.a} \quad . \tag{25}$$

The scalar integro-differential equation for $\tilde{\omega}_{ba}$ is

$$\left\{k_b^2 + D_b(r_\beta) - v_b^{eff}(r_\beta)\right\} \tilde{\omega}_{ba}(r_\beta) = 0 \quad . \tag{26}$$

Hence

$$\tilde{\omega}_{ba}(r_\beta) = \tilde{n}_{ba}\, w_b^{(o)}(r_\beta) + \frac{1}{k_b^2 + D_b(r_\beta) + i\varepsilon} v_b^{eff}(r_\beta)\, \tilde{\omega}_{ba}(r_\beta) \quad . \tag{27}$$

By use of a similar notation to that in eq. (22), the factorization of the a- and r_β -dependence may be expressed as

$$\tilde{\omega}_{ba}(r_\beta) = \tilde{\Omega}_b(r_\beta)\,\tilde{n}_{ba} \tag{28}$$

where, again, an explicit form for $\tilde{\Omega}$ follows from eq. (27).

An explicit expression for the mixed effective two-particle wave function u_{ba} is obtained by combining eq. (25) with eq. (28):

$$u_{ba}(r_\beta) = \tilde{\Omega}_b(r_\beta)\,\tilde{n}_{ba} + \frac{1}{k_b^2 + D_b(r_\beta) - v_b^{eff}(r_\beta) + i\varepsilon}\,\underline{v}^+_{.b}(r_\beta)\,\underline{\underline{\Omega}}(r_\beta)\underline{n}_{.a} \quad . \tag{29}$$

To complete the argument it is necessary to determine the vector and the scalar normalization factors

$$\underline{n}_{.a} \qquad \text{and} \qquad \tilde{n}_{ba} \tag{30}$$

that are functionals of the input channel wave function ϕ_a. This calculation must be based on an equation that relates the scattering quantities expressed in the variables of the output fragmentation to the input channel wave function, i.e.

$$\psi_a^+ = \frac{1}{E - \left\{T_\beta(r_\beta) + H_\beta(\xi_\beta)\right\} + i\varepsilon}\left\{i\varepsilon\,\phi_a + V_\beta\,\psi_a^+\right\} \tag{31}$$

which is the β-representation of the integral equation for scattering.

To put eq. (31) into a useful form we expand first the scattering state ψ_a^+ given by eq. (3), and subsequently also the expression $V_\beta\,\psi_a^+$, in terms of the channel wave functions b', which gives

$$V_\beta\,\psi_a^+ = r^{-1}\sum_{b''(\beta)}\left\{\sum_{b'(\beta)}\left[v_{b''b'}(r_\beta)\,u_{b'a}(r_\beta)\right]\phi_{b''}\left(\hat{\underline{r}}_\beta,\ \xi_\beta\right)\right\} \quad . \tag{32}$$

For the sake of simplicity, the bound and the continuous channels are treated on equal footing. A comparison of eqs. (3), (31) and (32) leads to a set $b''(\beta)$ of equations

$$u_{b''a}(r_\beta) = \int_0^\infty dr'_\beta \left\{ g_{b''}^{o(+)}(r_\beta, r'_\beta, k_{b''}) \right.$$

$$\left. \left[i\varepsilon\, f_{b''a}(r'_\beta) + \sum_{b'(\beta)} v_{b''b'}(r'_\beta)\, u_{b'a}(r'_\beta) \right] \right\} . \qquad (33)$$

Here, $g_b^{o(+)}$ is the force-free radial Green function of the outgoing type:

$$g_b^{o(+)}(r_\beta, r'_\beta, k_b) = k_b^2\, r_\beta\, r'_\beta\, j_{\ell_b}(k_b r_\beta)\, h^+_{\ell_b}(k_b r'_\beta) , \quad r_\beta < r'_\beta$$
$$= k_b^2\, r_\beta\, r'_\beta\, j_{\ell_b}(k_b r'_\beta)\, h^+_{\ell_b}(k, r_\beta) , \quad r > r'_\beta \; ; \qquad (34)$$

f_{ba} the overlap of the input and the output channels

$$f_{b''a}(r_\beta) \equiv \langle \phi_{b''} | \phi_a \rangle \qquad (35)$$

and the sum covers all the channels of the fragmentation β. Now, in order to have a set of equations for the quantities of (30), one has to substitute into eq. (33) both eq. (29) for u_{ba} and a combination of eqs. (29) and (19) for $\underline{u}_{.a}$.

It is worth noting that eq. (33) can be regarded as a set of coupled integral equations in a single variable for the complete set of $u_{b'a}$'s , which, in principle, can be solved, whenever the interaction operator as well as the input channel wave functions are given in the space of the output channels.

Once the set of effective two-particle state is known, the transition amplitude may be calculated, not only by eq. (1) but also by

$$T_{ba} = \sum_{b'(\beta)} \int dr_\beta\, r_\beta^{-2}\, w_b^{(o)}(r_\beta)\, v_{bb'}(r_\beta)\, u_{b'a}(r_\beta) \qquad (36)$$

which is an immediate consequence of expansion (32) and does not involve the channel radii.

In summarizing these considerations it should be pointed out that whereas the first of the above methods for calculating the effective two-particle state develops a set of equations based on the many-particle Schrödinger equation, the alternative procedure deduces a set of integral equations starting from the many-particle integral equation. While the latter method is self-contained, the first one requires information with regard to the formalism of the integral equation method.

Feshbach's theory has also been made the basis of some model calculations [4] that prove particularly successful in reproducing inelastic scattering experimental data. It is hoped that either of the procedures proposed above will find some application in model construction of rearrangement processes in few-nucleon problems or direct reactions.

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