KFKI-71-33

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CONSTRUCTION OF THE EFFECTIVE LAGRANGIAN OF PIONIC PROCESSES FOR ARBITRARY SU (2) x SU (2) BREAKING

Hungarian Academy of Sciences

CENTRAL
RESEARCH
INSTITUTE FOR
PHYSICS

BUDAPEST

2017

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ABSTRACT

By assuming that the SU(2)xSU(2) symmetry is broken by the isoscalar element of the representation (ℓ,ℓ) effective Lagrangians reproducing the results of current algebra and the PCAC assumption can be constructed by a direct method suggested by R. Dashen and M. Weinstein [2]. It is shown that the symmetry-breaking parts of these Lagrangians are the solutions /in closed form/ of the differential equation for the breaking parts in Weinberg's formalism [3], and thus the connection between the two approaches is established.

РЕЗЮМЕ

Предполагая, что изоскалярный элемент представления (ℓ,ℓ) нарушает симметрию SU(2)xSU(2), непосредственным методом, предложенным Р. Дашеном и М. Вейнштейном можно записать эффективные функции Лагранжа, которые приводят к тем же самым результатам как и в случае использования токовой алгебры и предположения PCAC. Показано, что нарушающие симметрию части этих функций Лагранжа представляют собой решения (в закрытом виде) дифференциального уравнения для нарушающих частей в формулировке Вейнберга [3], а также создана связь между двумя приближениями.

KIVONAT

Ha feltételezzük, hogy az (ℓ,ℓ) reprezentáció izoskalár eleme megtöri az SU(2)xSU(2) szimmetriát, akkor az R. Dashen és M. Weinstein [2] által javasolt direkt módszerrel előállithatók az effektiv Lagrange-függvények, amelyek ujból az áramalgebra és a PCAC-feltevés felhasználásával kapott eredményekhez vezetnek. Megmutatjuk, hogy e Lagrange-függvények szimmetria-törő része /zárt alakban/ a Weinberg-féle formalizmus [3] szimmetria-törő részére felirható differenciálegyenlet megoldását adja, és ezzel bebizonyitottuk, hogy a kétféle közelités között kapcsolat van.

I. INTRODUCTION

R. Dashen and M. Weinstein [1], [2] have pointed out that the most logical explanation for the success of the PCAC hypothesis is that the real world satisfies an approximate SU2xSU2 symmetry. The symmetry is realized by the appearance of massless pions /Goldstone bosons/, the vaccuum is not invariant.

Many general theorems have been proved in [2] with the help of an identity which gives the matrix element $\langle \alpha + n\pi | S | \beta \rangle$ in terms of matrix elements of time-ordered products of vector and axial vector currents. Starting from this identity the authors constructed in the SUxSU2 symmetry case the effective Lagrangian, which in the tree approximation reproduces the results of the joint assumptions of PCAC and current algebra.

Following this approach, I have looked for the effective Langrangian in the case of any type of symmetry breaking and any type of definition of the pion interpolating field. I shall discuss the connection with Weinberg's results [3], and thus complete the connection between the above mentioned definition of PCAC and Weinberg's phenomenological Lagrangian formalism. I shall then discuss how one can obtain the general form of the covariant derivative in this way, and prove that the symmetry-breaking parts of the Lagrangians are the solutions in closed form of the differential equation determining the breaking Lagrangian in Weinberg s formalism.

Current algebra, with the aid of the approximate symmetry assumption, determines the amplitudes on the mass shell, but at a non-physical point /whereas the usual PCAC technique determines the S matrix element off the mass shell/. I shall make some remarks about this in connection with the $\pi\pi$ amplitude.

II. THE EFFECTIVE LAGRANGIAN IN THE CASE OF ARBITRARY TYPE OF SYMMETRY BREAKING

The effective Lagrangians can be constructed in closed form for an arbitrary type of symmetry-breaking from one of Dashen and Weinstein's identities [2]. They are just the Weinberg's phenomenological Lagrangians in the case of general pion field definition. The mentioned identity is:

$$\langle \alpha + \pi_{\alpha_1}(p_1) + \pi_{\alpha_2}(p_2) + \ldots \pi_{\alpha_n}(p_n) | S | \beta \rangle = f_\pi^n \langle \alpha | U^{(n)}(p_1, p_2, \ldots p_n) | \beta \rangle \qquad /1/$$

with $U^{(n)}(p_1, p_2 \ldots p_n)$ defined to be the coefficient of f_π^n in the expansion of the exponential

$$T\left(\exp\left[i \int d^4x \, \mathcal{L}(x)\right]\right) \qquad /2/$$

In /2/ $\mathcal{L}(x)$ is in general a sum of two parts, which we can obtain from

$$f_\pi \int_0^1 du \left\{ \exp\left[-if_\pi u \int d^3x \, G(\varphi^2) \underline{\varphi} . \underline{A}^0(x)\right] \left(\partial_\mu \left(G(\varphi^2) \underline{\varphi}\right) \underline{A}^\mu(y)\right) \exp\left[if_\pi u \int d^3z \, G(\varphi^2) \underline{\varphi} . \underline{A}^0(z)\right]\right\} +$$

$$+ f_\pi \int_0^1 du \left\{ \exp\left[-if_\pi u \int d^3x \, G(\varphi^2) \underline{\varphi} . \underline{A}^0(x)\right] \left(G(\varphi^2) \underline{\varphi} . \partial_\mu \underline{A}^\mu(y)\right) \exp\left[if_\pi u \int d^3z \, G(\varphi^2) \underline{\varphi} . \underline{A}^0(z)\right]\right\} \qquad /3/$$

if in the resulting expression we replace the terms $\partial_\mu \underline{\varphi} . \underline{A}$, $\underline{\varphi} . \partial_\mu \underline{A}^\mu$ /linear in $\underline{\varphi}(x)$ / with $\partial_\mu \underline{\varphi} . \hat{\underline{A}}^\mu$, $\underline{\varphi} . \partial_\mu \widehat{\underline{A}}$. Here $\underline{\varphi}(x)$ is a c number isovector function:

$$\underline{\varphi}(x) = \sum_{j=1}^{n} \underline{\varepsilon}_j \, e^{ip_j x} \qquad /4/$$

where $\underline{\varepsilon}_j$ is an isovector all of whose components except its α_j -th component are equal to zero, and only those terms of the resulting expression should be kept for which all the p_j are distinct. $A_i^\mu(x)$ /i=1,2,3/ are the axialvector currents encountered in the theory of weak interactions, and the barred quantities are defined as being equal to the corresponding unbarred quantity with the pion pole removed. $G(\varphi^2)$ stands for an arbitrary function of φ^2.

Performing the operations in the first term of (3), we get

$$\frac{\left(\partial_\mu \underline{\varphi} . \underline{\varphi}\right)\left(\underline{\varphi} . \underline{A}^\mu\right)}{\varphi} K(\varphi) + \frac{\sin f_\pi G\varphi}{\varphi} \left(\partial_\mu \underline{\varphi} . \underline{A}^\mu\right) + \left(\partial_\mu \underline{\varphi} \times \underline{\varphi}\right) \underline{V}^\mu \frac{\cos f_\pi G\varphi - 1}{\varphi^2} \qquad /5/$$

where

$$K(\varphi) = f_\pi \frac{\partial G}{\partial \varphi} + f_\pi \frac{G}{\varphi} - \frac{\sin f_\pi G\varphi}{\varphi^2} \qquad /6/$$

Let $|\alpha\rangle = |\beta\rangle = |0\rangle$ in (1). In the lowest order we get contributions from direct and pole terms. Let us look for the contribution of the direct part. From the first term in (3) we obtain a part which is second order in pion momenta:

$$\left\{\langle 0|\ \frac{1}{2!}(-i)^2 \int T\big(\mathcal{L}(x)\ \mathcal{L}(y)\big)\, d^4x\ d^4y\ |0\rangle\right\}\Big|_{f_\pi^n} \qquad /7/$$

In the case of the direct diagram we can put a caret above the axial vector currents in (5), which gives

$$\Big\{\int d^4x\ d^4y\ \frac{-1}{2!}\left[\frac{(\partial_\mu \underline{\varphi}\cdot\underline{\varphi})\varphi_\alpha(x)}{\varphi} K(\varphi) + \frac{\sin f_\pi G\varphi}{\varphi}\ \partial_\mu \varphi_\alpha(x)\right]$$
$$\left[\frac{(\partial_\nu\underline{\varphi}\cdot\underline{\varphi})\varphi_\beta(y)}{\varphi} K(\varphi) + \frac{\sin f_\pi G\varphi}{\varphi}\ \partial_\nu \varphi_\beta(y)\right]\langle 0|T\big(\hat{A}^\mu_\alpha\ \hat{A}^\nu_\beta\big)|0\rangle\Big\}\Big|_{f_\pi^n} \qquad /8/$$

Here

$$\langle 0|T\big(\hat{A}^\mu_\alpha\ \hat{A}^\nu_\beta\big)|0\rangle \longrightarrow \delta_{\alpha\beta}\ g^{\mu\nu}\ \frac{-i}{4f_\pi^2}$$

Clearly we obtain the same result by putting $\underline{\varphi}(x) \longrightarrow \underline{\pi}(x)$ in (8), sandwiching it between n pions, and calculating the direct diagram. Thus the kinetic part of the effective Lagrangian is

$$-\frac{1}{2}\, D_\mu\ \underline{\pi}(x)\ D^\mu \underline{\pi}(x) \qquad /9/$$

where

$$D_\mu \pi \equiv \frac{\sin f_\pi G\pi}{\pi}\ \partial_\mu \underline{\pi} + \frac{(\partial_\mu \underline{\pi}\cdot\underline{\pi})\underline{\pi}}{\pi} K(\pi) \qquad \pi \equiv \sqrt{\underline{\pi}^2} \qquad /10/$$

Let us compare this with the general form of the covariant derivative of Weinberg [3] *

* There is a misprint in the expression of $D_\mu \underline{\pi}$ and $V(\pi^2)$ in [3]

$$D_\mu \underline{\pi} = \frac{1}{\left(f^2(\pi^2)+\pi^2\right)^{1/2}} \partial_\mu \underline{\pi} + \frac{1}{f^2(\pi^2)+\pi^2} \left(f'(\pi^2) + \frac{1}{2}v\right)\underline{\pi}.\partial_\mu \underline{\pi}^2 \qquad /11/$$

where

$$v(\pi^2) = - \frac{f(\pi^2) - \left(f^2(\pi^2)+\pi^2\right)^{1/2}}{\pi^2} \; ; \quad f'(\pi^2) \equiv \frac{df(\pi^2)}{d\pi^2}$$

It can be seen immediately that the two expressions are the same, assuming

$$\sin f_\pi G\pi = \frac{\pi}{\left(f^2(\pi^2) + \pi^2\right)^{1/2}} \qquad /12/$$

To find the symmetry-breaking part of the Lagrangian assume that the strong interaction Hamiltonian can be decomposed at any t time into two parts

$$H = H_o(t) + \epsilon H_1(t) \; ; \; H_1(t) = \int \mathcal{H}_1(x) d^3x$$

where $H_o(t)$ possesses a chiral symmetry, and $\epsilon H_1(t)$ transforms as some sum of irreducible tensors under the SU2xSU2 group. The first order term in ϵ comes from

$$\langle 0| \frac{1}{1!}(-i) \int d^4x \, \mathcal{L}(x) |0\rangle \qquad /13/$$

Its value can be obtained by evaluating the second part of (3). Here we must make a general chiral transformation on the $\partial_\mu \underline{A}^\mu(y)$ operator, assuming that the symmetry-breaking operator is the isoscalar element of the representation (ℓ,ℓ) . Isoscalar and isovector ... elements arise from the transformation, but because of (13) we must look only for the coefficient of isoscalar part. Taking into account that

$$\langle 0| \epsilon \, \mathcal{H}_1^{(\ell,\ell)}(0) |0\rangle = - \frac{m_\pi^2}{4f_\pi^2} \frac{3}{2\ell(2\ell+2)} \qquad /14/$$

one can see that the symmetry-breaking part of the effective Lagrangian is

$$\left\{ \mathcal{L}^{(\ell,\ell)}(x) = -if_\pi \, G(\varphi^2) \frac{m_\pi^2}{4f_\pi^2} \frac{3}{\ell(2\ell+1)(2\ell+2)} \int_0^1 du \, \mathrm{Tr} \left\{ \underline{\varphi} \cdot \underline{J}^{(\ell)} . D^{(\ell)} \left(\underline{n}, 2\omega \right) \right\} \right|_{\underline{\varphi}(x) \to \underline{\pi}(x)} \qquad /15/$$

In /15/ $\underline{J}^{(\ell)}$ denotes the representation of the SU2 generator and $D^{(\ell)}(\underline{n},2\omega)$ the matrix for the rotation $2\omega = 2\left(-\varphi f_\pi Gu\right)$ about the axis $\underline{n} = \frac{\underline{\varphi}}{\varphi}$ of $(2\ell+1)$ dimensions /see Appendix/.

We find from the known [4] form of $D^{(\ell)}$

$$\mathcal{L}^{(\ell,\ell)}(x) = \frac{m_\pi^2}{4f_\pi^2} \frac{3}{\ell(2\ell+1)(2\ell+2)} \sum_{n=1}^{\ell} \left[\cos\left(2nf_\pi\, G\varphi\right) - 1\right]_{|\underline{\varphi}(x)\to\underline{\pi}(x)} \qquad /16/$$

/n=1/2 if ℓ is halfinteger/

With the aid of /12/ we get

$$\mathcal{L}^{(\ell,\ell)}(x) = \text{const} \,.\, \mathrm{Re}\left\{\frac{q^{\ell+1}}{q-1}\right\} + \text{const}' \qquad /17/$$

where $q = \frac{f(\pi^2)+i\pi}{f(\pi^2)-i\pi}$, and the constants may be obtained by demanding that the Taylor series of $\mathcal{L}^{(\ell,\ell)}(\pi^2)$ start with $\frac{1}{2}\, m_\pi^2\, \pi^2$.

Let us summarize our results in the following theorem:

In the lowest order of momenta and ϵ the pion's amplitudes are correctly calculated by using the effective Lagrangian /in the case of (ℓ,ℓ) -type symmetry-breaking/

$$\mathcal{L}(x) = -\frac{1}{2} D_\mu\underline{\pi} \,.\, D^\mu\underline{\pi} + \mathcal{L}^{(\ell,\ell)}(x) \qquad /18/$$

where $D_\mu\pi(x)$, $L^{(\ell,\ell)}(x)$ are defined by (10) / or (11) / and (16) /or (17)/ respectively, calculating according to the usual Feynman rules but subject to the restriction that one keeps only tree diagrams. In this way the connection between the point of view of broken symmetry and Weinberg's phenomenological Lagrangian formalism becomes complete.

Namely, $\mathcal{L}^{(\ell,\ell)}(x)$ is the solution of the differential equation:

$$2\pi^2\left(f(\pi^2)+\pi^2 g(\pi^2)\right)^2 \mathcal{L}^{(\ell,\ell)''}(\pi^2) +$$

$$+\left(f(\pi^2)+\pi^2 g(\pi^2)\right)\left[3f(\pi^2)+\pi^2 g(\pi^2)+2\pi^2\left(f(\pi^2)+\pi^2 g(\pi^2)\right)'\right]\mathcal{L}^{(\ell,\ell)'}(\pi^2)+\ell(2\ell+2)\,\mathcal{L}^{(\ell,\ell)}(\pi^2)=$$

$= \text{const.}$

$$g(\pi^2) = \frac{1 + 2f(\pi^2)f'(\pi^2)}{f(\pi^2) - 2\pi^2\, f'(\pi^2)} \qquad /19/$$

which gives the symmetry-breaking Lagrangian in Weinberg's method, as we can convince ourselves by direct substitution. The other solutions of (19) are $\sim \mathrm{Im}\left(\frac{q^{\ell+1}}{q-1}\right)$, which are singular when $\pi^2=0$, and we disregard them

In the special case of $f(\pi^2) = -\frac{1}{2f_\pi}\left(1 - f_\pi^2\,\pi^2\right)$ we get:

$$\mathcal{L}^{(1/2,1/2)}(\pi^2) = \frac{m_\pi^2}{2}\,\pi^2\,\frac{1}{1+f_\pi^2\,\pi^2}\,, \qquad /20a/$$

$$\mathcal{L}^{(1,1)}(\pi^2) = \frac{m_\pi^2}{2}\,\pi^2\,\frac{1}{\left(1+f_\pi^2\pi^2\right)^2}\,, \qquad /20b/$$

$$\mathcal{L}^{(2,2)}(\pi^2) = \frac{m_\pi^2}{2}\,\pi^2\,\frac{1 - 6/5 f_\pi^2\pi^2 + f_\pi^4\pi^4}{\left(1+f_\pi^2\pi^2\right)^4} \qquad /20c/$$

$20/_a$ is the only closed solution which was demonstrated by Weinberg [3].

III. THE $\pi\pi$ AMPLITUDE

The usual method gives the off-mass shell amplitude as [5]:

$$\langle\pi_\gamma(p_1)\pi_\alpha(q)|S|\pi_\beta(k)\pi_\delta(p_2)\rangle\Big|_{\substack{q^2=k^2=0\\ s=m_\pi^2+2p_1q\\ t=0\\ u=m_\pi^2-2p_1q}} = -i(2\pi)^4\delta^4(p_1+q-p_2-k)\,4f_\pi^2\cdot$$

$$\left[2(p_1q)\left(\delta_{\alpha\delta}\delta_{\gamma\beta} - \delta_{\alpha\gamma}\delta_{\delta\beta}\right) + i\langle\pi_\gamma(p_1)|\Sigma_{\alpha\beta}(q)|\pi_\delta(p_2)\rangle\right] \qquad /21/$$

where

$$\Sigma_{\alpha\beta}(q) = \int d^4x\, e^{iqx}\,\delta(x_o)\left[A_\alpha^o(x),\ \partial_\mu A_\beta^\mu(0)\right] \qquad /22/$$

With regard to the Adler self-consistency condition, Bose statistics and crossing symmetry, if $\Sigma_{\alpha\beta}\sim\delta_{\alpha\beta}$ /as it is in the σ model/, we get

$$M_1 = -4f_\pi^2\left(t - m_\pi^2\right) \quad M_2 = -4f_\pi^2\left(u - m_\pi^2\right) \quad M_3 = -4f_\pi^2\left(s - m_\pi^2\right) \qquad /23/$$

/about the point $s=u=m_\pi^2$, $t = 0$ /

Extrapolating this up to the threshold /here the PCAC assumption enters into the game/ we obtain the known scattering lengths.

On the other hand, with the broken symmetry method we get the amplitude on the mass shell. Indeed, let us consider the expression

$$\langle\pi_\gamma(p_1)|W_{\alpha\beta}|\pi_\delta(p_2)\rangle \equiv$$

$$\equiv\langle\pi_\gamma(p_1)|\Big\{T\big(\partial_\alpha(q)\partial_\beta(-k)\big)+(iq_\mu)T\big(A^\mu_\alpha(q)\partial_\beta(-k)\big)+(-ik_\nu)T\big(\partial_\alpha(q)A^\nu_\beta(-k)\big)+$$
$$+(iq_\mu)(-ik_\nu)\,T\big(A^\mu_\alpha(q)\,A^\nu_\beta(-k)\big)\Big\}|\pi_\delta(p_2)\rangle$$

$$T\big(\partial_\alpha(q)\,\partial_\beta(-k)\big) \equiv \int d^4x\,d^4y\,e^{iqx}\,e^{-iky}\,T\big(\partial_\mu A^\mu_\alpha(x)\,\partial_\nu A^\nu_\beta(y)\big) \text{ and so on} \qquad /24/$$

Pulling all derivatives through the time-ordered instruction, separating off all terms corresponding to the pion-pole diagrams, we get in lowest order of the momenta q, k and of the symmetry-breaking parameter ϵ :

$$\langle\pi_\gamma(p_1)\pi_\alpha(q)|S|\pi_\beta(k)\pi_\delta(p_2)\rangle\left(q^2=m_\pi^2,\ k^2=m_\pi^2,\ \xi,\eta\right)\approx -i(2\pi)^4\,\delta^4(p_1+q-p_2-k)$$

$$4f_\pi^2\Big[2(p,q)\big(\delta_{\alpha\delta}\delta_{\gamma\beta}-\delta_{\alpha\gamma}\delta_{\delta\beta}\big)-i\langle\pi_\gamma(p_1)|\Sigma_{\alpha\beta}(q)|\pi_\delta(p_2)\rangle\Big] \qquad /25/$$

Here ξ , η are unspecialized variables which were fixed when we extracted the pion-pole terms. Let us compare this with expression (21). We observe that there the sign of the Σ commutator is opposite, the two expressions are different. However, this is understandable, since they determine the S matrix element at different values of their variables. Of course, the Adler selfconsistency condition, which is an off-mass shell statement, does not make any restriction on (25). On the other hand, we can determine the Σ commutator's matrix element in our approximation. Let us consider the identity

$$\int d^4x\,e^{ipx}\,T\big(\partial_\gamma(x)\epsilon\mathcal{H}_1(o)\big)=-(ip_\mu)\int d^4x\,e^{ipx}\,T\big(A^\mu_\gamma(x)\epsilon\mathcal{H}_1(o)\big)-\int d^4x\,e^{ipx}\,\delta(x_o)\big[A^o_\gamma(x),\epsilon\mathcal{H}_1(o)\big] \qquad /26/$$

Sandwiching it between $\langle o|\ldots|\pi_\delta(p_2)\rangle$, separating off the pion-pole terms, we get in the case of $p\to o$ and in first order of ϵ :

$$\langle\pi_\gamma(p_1)|\Sigma_{\alpha\beta}(q)|\pi_\delta(p_2)\rangle_{p_1^2=p_2^2=m_\pi^2}=-i\delta_{\alpha\beta}\,\langle\pi_\gamma(p_1)|\epsilon\mathcal{H}_1(o)|\pi_\delta(p_2)\rangle_{p_1^2=p_2^2=m_\pi^2}=$$
$$=-i\delta_{\alpha\beta}\,\delta_{\gamma\delta}\cdot m_\pi^2$$

The left-hand side of (25) is independent from q^2, k^2 in the case of any ξ , η . This property must also hold after the approximation, and this gives a restriction on ξ , η . If we choose $\xi\equiv\nu=\frac{1}{2}(q+k)\cdot(p_1+p_2)$ $\eta\equiv x=(qk)$, then with the aid of (25), (27).

$$M_1\left(q^2=k^2=m_\pi^2,\ \nu,\ x=o\right)\approx -4f_\pi^2\,m_\pi^2$$

$$M_2\left(q^2=k^2=m_\pi^2,\ \nu,\ x=o\right)\approx -4f_\pi^2(-\nu) \qquad /28/$$

$$M_3\left(q^2=k^2=m_\pi^2,\ \nu,\ x=o\right) \approx -4f_\pi^2\ \nu \qquad /28/$$

Since $q^2=k^2=m^2$, $x=0$, therefore $s=m_\pi^2+\nu$, $u=m_\pi^2-\nu$, $t=2m^2$. Because of Bose statistics and crossing-symmetry, we find from (28) that around the on-mass shell point $s=u=m_\pi^2$, $t=2m_\pi^2$

$$M_1 \approx -4f_\pi^2\left(t-m_\pi^2\right) \qquad M_2 \approx -4f_\pi^2\left(u-m_\pi^2\right) \qquad M_3 \approx -4f_\pi^2\left(s-m_\pi^2\right) \qquad /29/$$

which is identical with Weinberg's result extrapolated to the mass shell.

APPENDIX

If x_i and y_i $/i=-\ell,..0,..\ell/$ are the representations of the two independent SU2 generators, then

$$\mathcal{H}_1^{(\ell,\ell)} = x_k\, y^k \qquad \text{A.1}$$

Let us begin with the second term of (3). Because

$$\partial_\mu \underline{A}^\mu(x) = -i\left[\underline{F}^5,\ \epsilon\, \mathcal{H}_1^{(\ell,\ell)}(x)\right] \qquad \text{A.2}$$

we get

$$-i\epsilon f_\pi\ G(\varphi^2)\int_0^1 du\left[\underline{\varphi}(y)\,\underline{F}^5, e^{-iuf_\pi G(\varphi^2)\,\underline{\varphi}(y)\underline{F}^5}\,\mathcal{H}_1^{(\ell,\ell)}(y)\, e^{iuf_\pi G(\varphi^2)\,\underline{\varphi}(y)\underline{F}^5}\right] \qquad \text{A.3}$$

where the $\delta^3(\underline{x}-\underline{y})$ appearing in the equal time commutators has been used to exchange $\underline{A}^o \to \underline{F}^5$. With the aid of A.1

$$e^{-iuf_\pi G\underline{\varphi}.\underline{F}^5}\,\mathcal{H}_1^{(\ell,\ell)}\, e^{iuf_\pi G\underline{\varphi}.\underline{F}^5} = \left(e^{-is\underline{\varphi}.\underline{F}^{(+)}}\, x_j\, e^{is\underline{\varphi}.\underline{F}^{(+)}}\right)\left(e^{is\underline{\varphi}.\underline{F}^{(-)}}\, y^j\, e^{-is\underline{\varphi}.\underline{F}^{(-)}}\right)$$

$$= x_k\left(D^{(\ell)}(\underline{n},\omega)\ D^{(\ell)}(\underline{n},\omega)\right)_{kt} y^t = x_k\ D^\ell(\underline{n},2\omega)_{kt}\ y^t \qquad \text{A.5}$$

$$/s = f_\pi\ Gu/$$

where $D^\ell(\underline{n},\omega)$ is the matrix of the rotation $\omega=-s\varphi$ about the axis $\underline{n} = \frac{\underline{\varphi}}{\varphi}$ in $(2\ell+1)$ dimensions. Let us substitute A.5 in A.3,

$$-i\epsilon f_\pi G\int_0^1 du\ x_k\left\{\underline{\varphi}\ \underline{J}^{(\ell)},\ D^{(\ell)}(\underline{n},2\omega)\right\}_{+kt}\ y^t \qquad \text{A.6}$$

denoting by $\overline{J}^{(\ell)}$ the representation of the SU2 generator. We must look for the isoscalar element from A.6, because of (13). It is the trace times $\frac{1}{2\ell+1}$. On the other hand

$$<0|\,\epsilon\,\mathcal{H}_1^{(\ell,\ell)}(0)|0> = -\frac{m_\pi^2}{4f_\pi^2} - \frac{3}{2\ell(2\ell+2)}$$

and so we obtain

$$\mathcal{L}^{(\ell,\ell)}(x) = -if_\pi G(\varphi^2)\,\frac{m_\pi^2}{4f_\pi^2}\,\frac{3}{\ell(2\ell+2)(2\ell+1)}\int_0^1 du\;\mathrm{Tr}\left(\underline{\varphi}\,\underline{J}^{(\ell)}D^{(\ell)}(\underline{n},2\omega)\right)\Big|_{\underline{\varphi}\to\underline{\pi}(x)} \qquad \text{A.8}$$

The relation between $\underline{n}$, 2ω and the appropriate Euler angles is

$$\begin{bmatrix} \cos\frac{\Theta}{2}\,e^{i\frac{\phi_1+\phi_2}{2}} & i\sin\frac{\Theta}{2}\,e^{-i\frac{\phi_2-\phi_1}{2}} \\ i\sin\frac{\Theta}{2}\,e^{i\frac{\phi_2-\phi_1}{2}} & \cos\frac{\Theta}{2}\,e^{-i\frac{\phi_1+\phi_2}{2}} \end{bmatrix} \leftrightarrow \begin{bmatrix} \cos\omega - i\frac{\varphi_3}{\varphi}\sin\omega & i(\varphi_1+i\varphi_2)\frac{1}{\varphi}\sin\omega \\ i(\varphi_1-i\varphi_2)\frac{1}{\varphi}\sin\omega & \cos\omega+i\frac{\varphi_3}{\varphi}\sin\omega \end{bmatrix} \qquad \text{A.9}$$

The expression A.8 is an isoscalar, so we can proceed in a special coordinate system: $\underline{\varphi} = (0,0,\varphi) \rightarrow \Theta=0$. Then it is easy to see that

$$\mathrm{Tr}\left(\underline{\varphi}\,\underline{J}^{(\ell)}D^{(\ell)}(\phi_1,\Theta,\phi_2)\right) = \sum_{n=-\ell}^{\ell} n.\varphi\, e^{-in(\phi_1+\phi_2)} \qquad \text{A.10}$$

By A.8 and A.9 we get (16). On the other hand

$$\mathcal{L}^{(\ell,\ell)} \sim \sum_{\substack{n=1\\(n=1/2)}}^{\ell} \cos(2nf_\pi\, G\varphi)\Big|_{\underline{\varphi}\to\underline{\pi}} = \sum_{\substack{n=1\\(n=1/2)}}^{\ell} \mathrm{Re}\left(\cos f_\pi G\varphi + i\sin f_\pi G\varphi\right)^{2n}\Big|_{\underline{\varphi}\to\underline{\pi}} \qquad \text{A.11}$$

From (12) we get

$$\left(\cos f_\pi G\varphi + i\sin f_\pi G\varphi\right)^{2n} = \left(\frac{f+i\varphi}{f-i\varphi}\right)^n \equiv q^n \qquad \text{A.12}$$

Substituting A.12 in A.11 we obtain (17), because

$$\mathrm{Re}\,\frac{q}{q-1} = \mathrm{const.} \qquad \text{A.13}$$

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