KFKI 2/1963

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DETERMINATION OF THE PROMPT NEUTRON PERIOD FROM THE FLUCTUATIONS OF THE NUMBER OF NEUTRONS IN A REACTOR

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DETERMINATION OF THE PROMPT NEUTRON PERIOD FROM THE FLUCTUATIONS OF THE NUMBER OF NEUTRONS IN A REACTOR

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Summary

A detailed mathematical analysis is given of the prompt neutron period measurement based on the fluctuation of the number of neutrons in a reactor. In the case of the Feynman method the effect of the delayed neutrons on the accuracy of the measurement is investigated and the correlation between the numbers of neutrons counted in consecutive time intervals is examined so as to make possible appropriate choice of the conditions of measuring. For the Rossi method a general theory is developed taking into account the role of the delayed neutrons. It was found that the Rossi method can be used in the case of thermal reactors provided the average neutron lifetime is not larger than about 10^{-4} sec. Moguilner's "Zero Probability Method" is discussed and the theoretical basis of this method is given. Last, a new method /differential method/ is proposed which is less sensitive to the correlation than the Feynman method. The Rossi and Moguilner methods proved to be faster and more reliable than the other two methods.

Introduction

Safety considerations draw attention in particular to two of the parameters determining the kinetic behaviour of reactors, namely the mean prompt neutron lifetime (ℓ) and the effective fraction of delayed neutrons β^{eff}.

The mean lifetime of the prompt neutrons depends on their initial energy and on the structure and size of the reactor. The mean number of delayed neutrons per fission is known from direct measurements [1]. Since the delayed neutrons have lower energies than the prompt neutrons, their effectiveness in the chain reaction is expected to be different from that of the prompt neutrons. This difference in effectiveness is usually taken into account by using a factor for modifying the mean number of delayed neutrons per fission and instead of the values $\beta_i = \nu_i/\nu$ $(i=1,\ldots,6)$ the values $\beta_i^{eff} = e_i\beta_i$ $(i=1,\ldots,6)$ are used. /Here ν_i is the mean number of delayed neutrons of i-th group per fission and ν is the total number of neutrons per fission./

It is of interest to note that for small thermal reactors the effectiveness of the delayed neutrons is higher than that of the prompt neutrons, since the delayed neutrons are less likely to leak from the reactor that the prompt neutrons. On the other hand, in reactors where the number of fissions initiated by high energy neutrons is rather high, the effectiveness of the prompt neutrons becomes higher.

Since the experimental values of β_i^{eff} $(i=1,\ldots,6)$ are not at present available we shall assume that $e_i = e$ $(i=1,\ldots,6)$. In this case $\beta^{eff} = e\sum_{i=1}^{6} \beta_i = e\beta$ and this is the quantity that plays an important role in reactor dynamics calculations.

Methods have been developed for calculating the values both of ℓ and β^{eff} [2], these theoretical values, however, are still to be verified by reliable experimental methods. The experimental check of the calculated values seems to be of particular interest in small thermal reactors moderated by hydrogeneous materials in which the values of ℓ and β^{eff} are sensitive to composition and geometry. The measurements of either β^{eff} or ℓ present considerable difficulties, however, the ratio β^{eff}/ℓ may be determined through a relatively simple measurement.

For the experimental determination of the ratio β^{eff}/ℓ namely the random fluctuations of the neutron multiplication can be utilized. The relaxation constants characterizing the dynamic behaviour of a reactor are determined by the roots $\omega_0 < \omega_1 < \cdots < \omega_5 < \omega_6$ of the equation

$$\frac{\beta^{eff}}{\ell} - \frac{k-1}{k\ell} - \omega = \frac{\beta^{eff}}{\ell\beta} \sum_{i=1}^{6} \frac{\lambda_i \beta_i}{\lambda_i - \omega} ,$$

where k is the multiplication factor and λ_i the decay constant for nuclei emitting delayed neutrons of type i. It is of interest to note that only the largest of the roots (ω_6) is sensitively dependent on the neutron lifetime. The largest relaxation constant is about equal to the quantity

$$\alpha = \frac{\beta^{eff}}{\ell} - \frac{k-1}{k\ell} = \frac{\beta^{eff}}{\ell} - \frac{\rho}{\ell} ,$$

the reciprocal value of which gives the prompt neutron period.

Since it is reasonable to expect that the short-term behaviour of the fluctuations depends sensitively on the largest relaxation constant

ω_6 only, the analysis of rapid fluctuations seems to be the most promising method for the determination of ω_6 . The values of ω_6 , as measured at various reactivities can be used to calculate the ratio β^{eff}/ℓ .

The actual task therefore is to determine the largest relaxation constant $\omega_6 \sim \alpha$ at various negative reactivities. There are several methods for doing this, our considerations, however, will be restricted to the 1/ direct /Feynman/, 2/ correlation /Rossi/, 3/ zero probability /Moguilner/ and 4/ differential methods.

Direct method

The variance of the number of counts recorded during the time Δt by a neutron detector placed into a steadily operating subcritical system containing a neutron source can be expressed in the form

$$D(\Delta t) = N\Delta t\left[1+\varepsilon\sum_{j=0}^{6} D_j\psi_j(\Delta t)\right] = N\Delta t\,(1+\varepsilon\phi) \, , \qquad /1.1/$$

where $N\Delta t$ is the expected number of counts recorded during the time Δt, ε is the detector efficiency, D_j is a constant independent of Δt , while

$$\psi_j(\Delta t) = 1 - \frac{1-e^{-\omega_j \Delta t}}{\omega_j \Delta t} \, . \qquad /1.2/$$

The variance /1.1/ has been derived by several authors [3], [4]. In the following we are going to use the results of [4]. The dependence of ϕ on the counting interval Δt at various reactivities can be seen in Fig.1.

It is of interest to determine the constributions of the different terms $\phi_j = D_j\psi_j$ $(j = 0,\dots,6)$ to the quantity ϕ for various

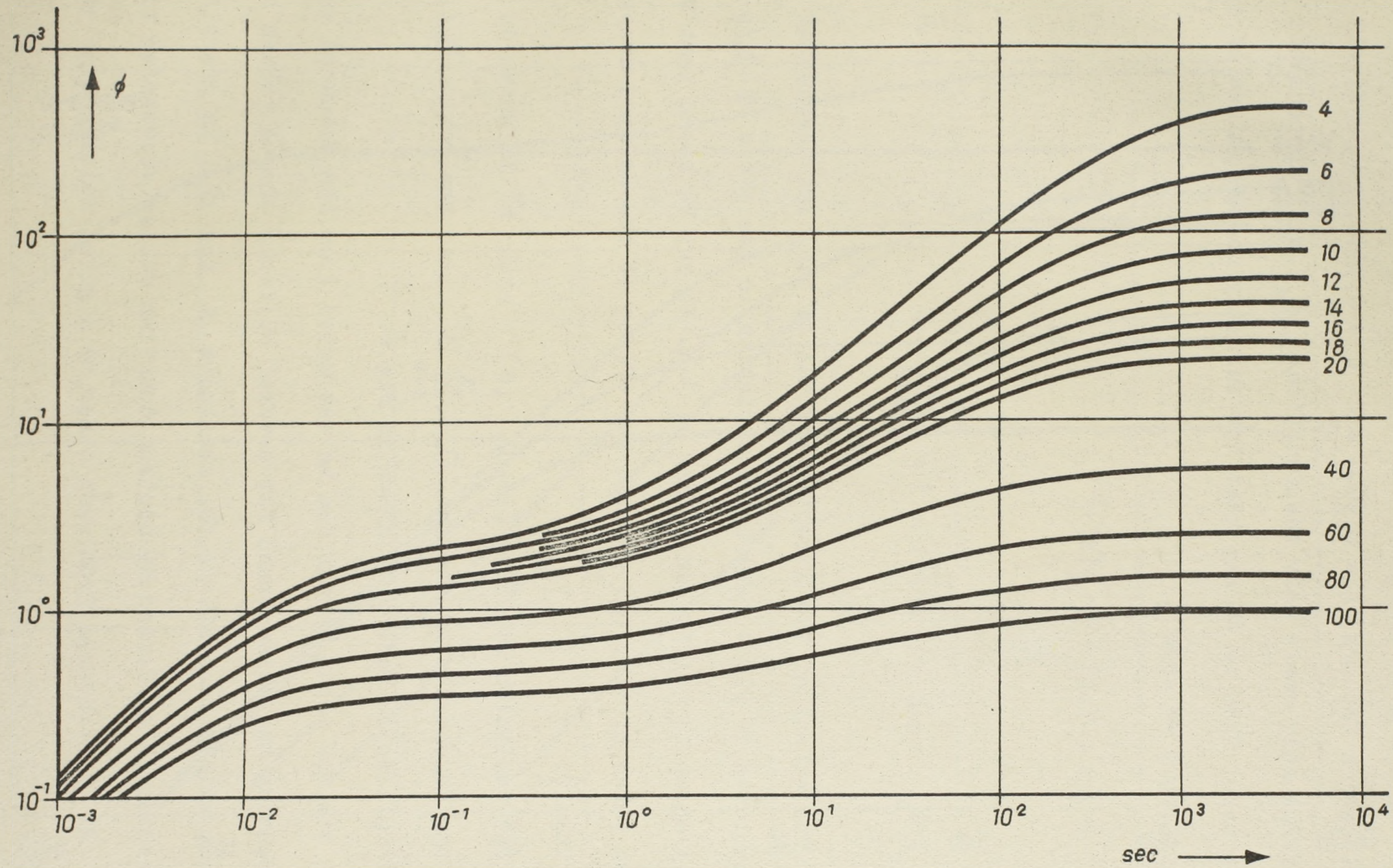


Fig.1 Dependence of Φ on the counting interval Δt for different values of $1-k$ in 10^{-4} units ($\tau = 5.10^{-5}$ sec)

Δt -s. In Figs. 2 and 3 the time dependence of each ϕ_j is plotted for the multiplication factors $k = 0{,}9900$ and $k = 0{,}9990$, respectively. If Δt is sufficiently small only the function ϕ_6 will appreciably differ from zero. That is to say that for suitably chosen values of the counting intervals we have

$$D(\Delta t) \sim N\Delta t\left[1 + a\,\psi(\Delta t)\right] \quad , \qquad /1.3/$$

where

$$a = \varepsilon D_6 \qquad \text{and} \qquad \psi(\Delta t) = 1 - \frac{1 - e^{-\alpha\Delta t}}{\alpha\Delta t} \quad . \qquad /1.4/$$

The effect of the delayed neutrons on the time dependence of ϕ is illustrated in Fig.4, where the variation in the ratio $\phi - \phi_6/\phi = \delta/\phi$ with Δt is shown for different values of $1-k$. It may be noted that the smaller the value of $1-k$, the earlier the effect of the delayed neutrons manifests itself.

The variation of D_j $(j = 0, \ldots, 6)$ with k is shown in Fig.5 and the dependence of the sum $D_t = \sum_{j=0}^{6} D_j$ and the ratio D_6 / D_t on the multiplication factor is shown in Fig.6.

Now the ratio $D/N\Delta t$ can be estimated from the measured values and by repeating the measurement for several suitably chosen values of Δt, it is possible to evaluate the constants α and a. In practice, the measurement is performed by neutron counting during successive time intervals Δt, separated from each other by a given waiting time Θ /See Fig.7./

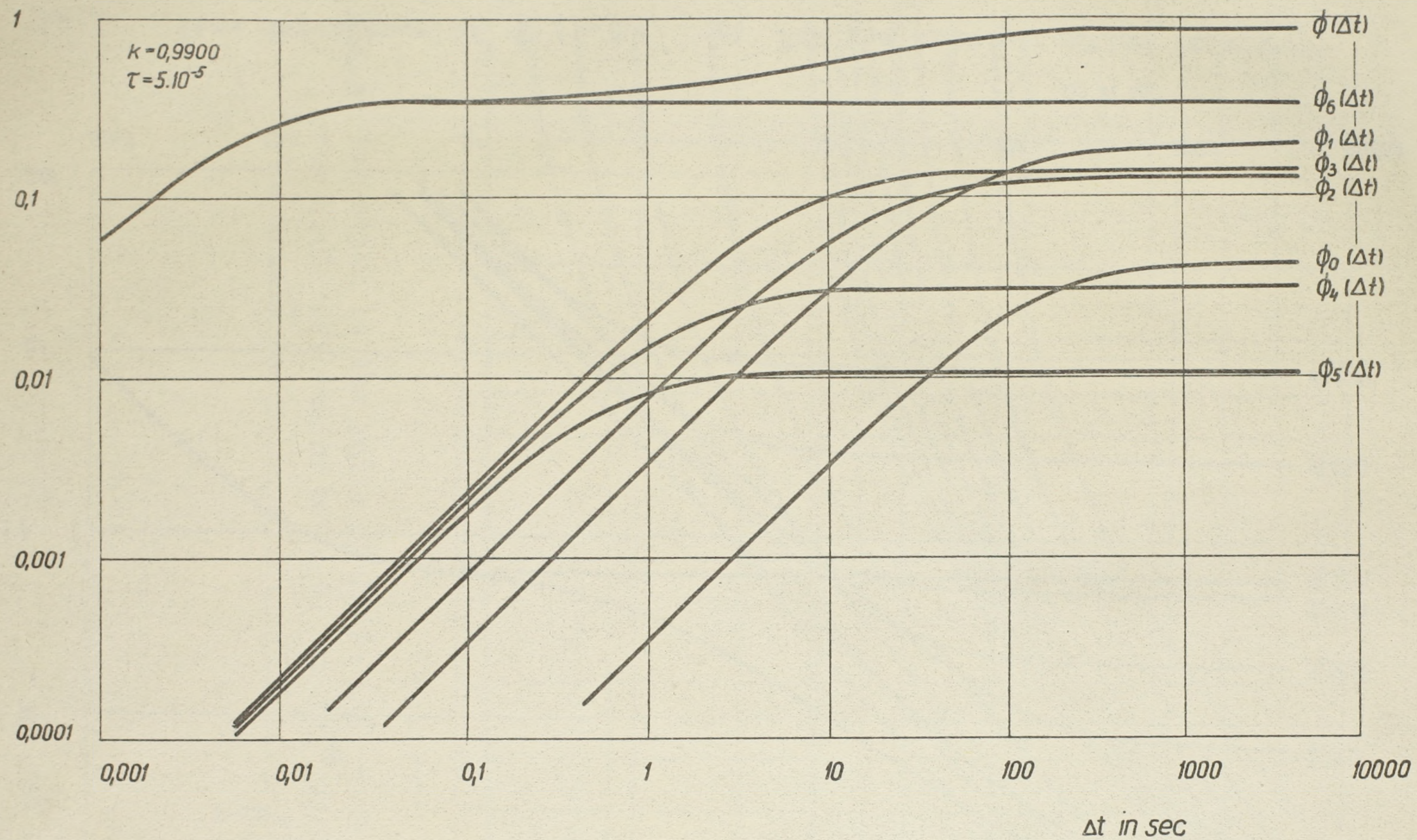


Fig.2 Time dependence of Φ_j ($j = 0, 1, \ldots, 6$) for the multiplication factor k = 0,9900

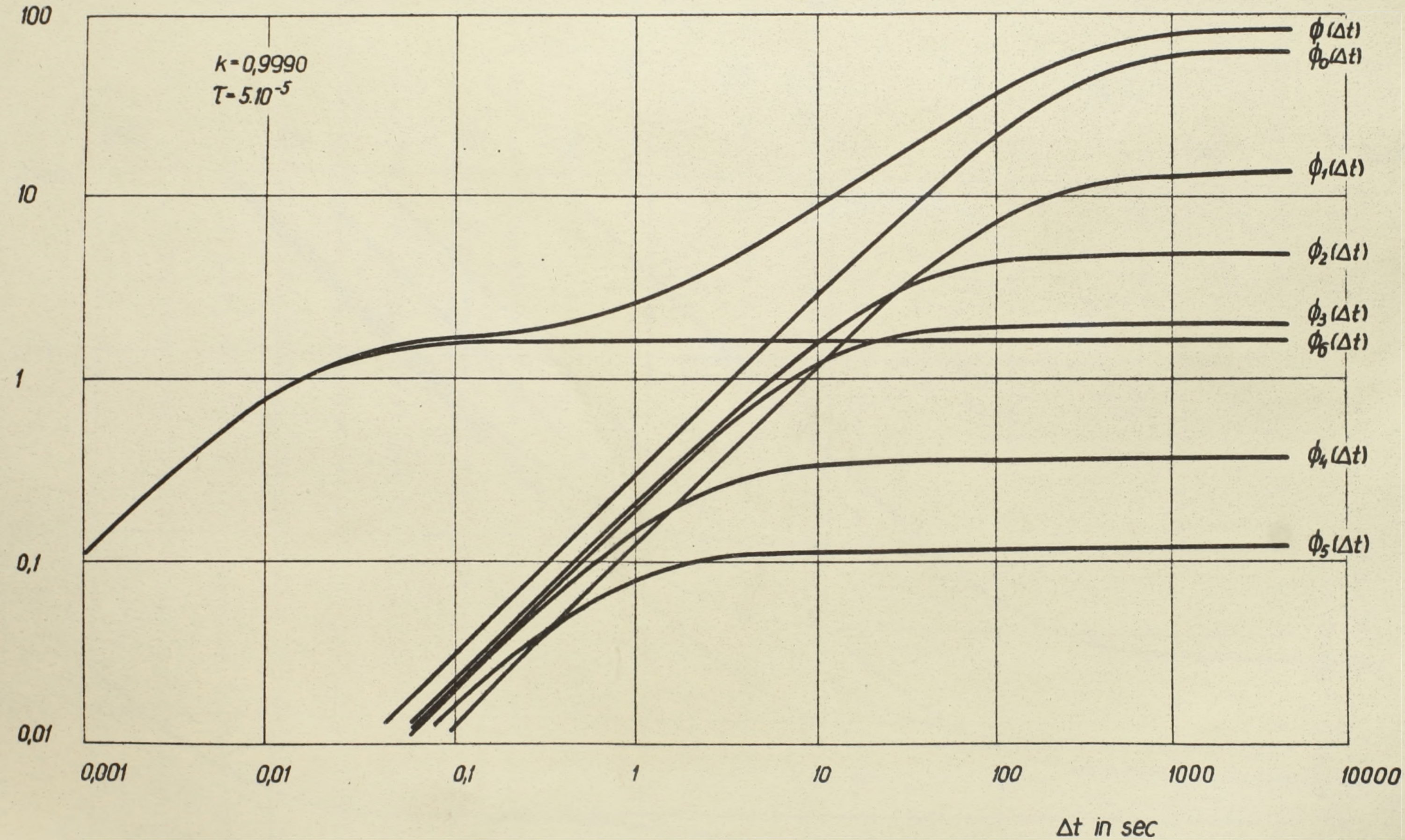


Fig.3 Time dependence of Φ_j $(j = 0, 1, \dots, 6)$ for the multiplication factor $k = 0,9990$

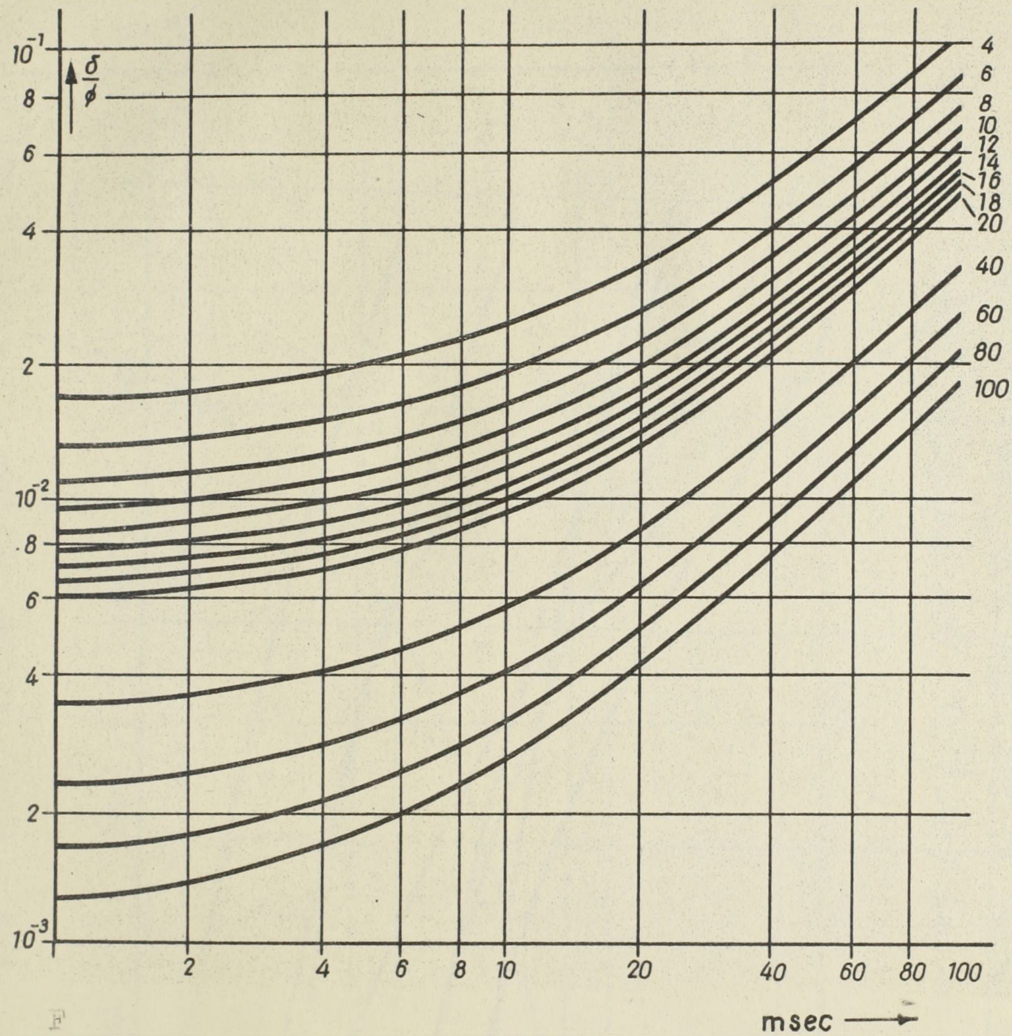


Fig.4 Variation in the ratio $(\Phi-\Phi_s)/\Phi = \delta/\Phi$ with for different values of $1-k$ in 10^{-4} units ($\tau = 5.10^{-5}$ sec)

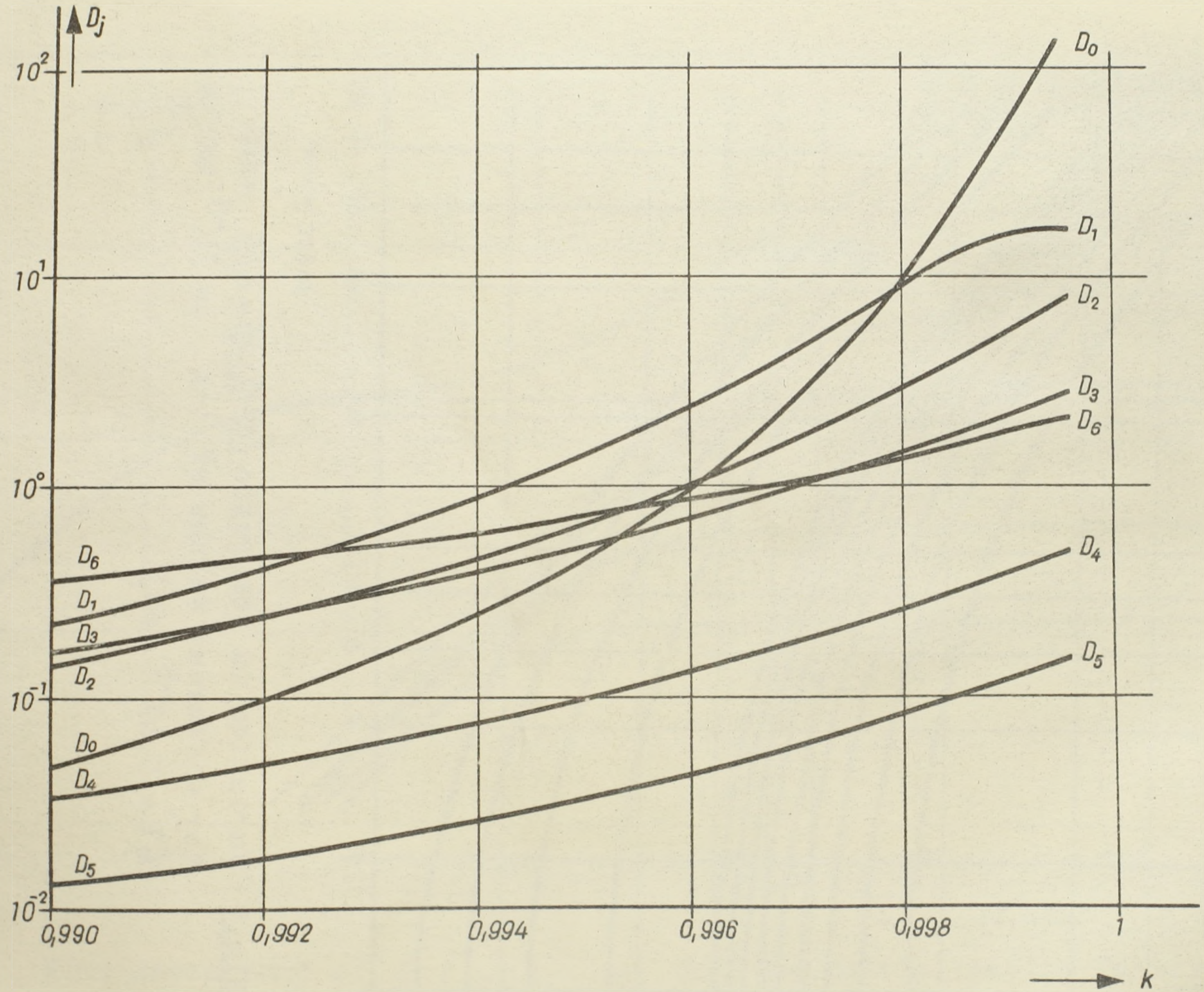
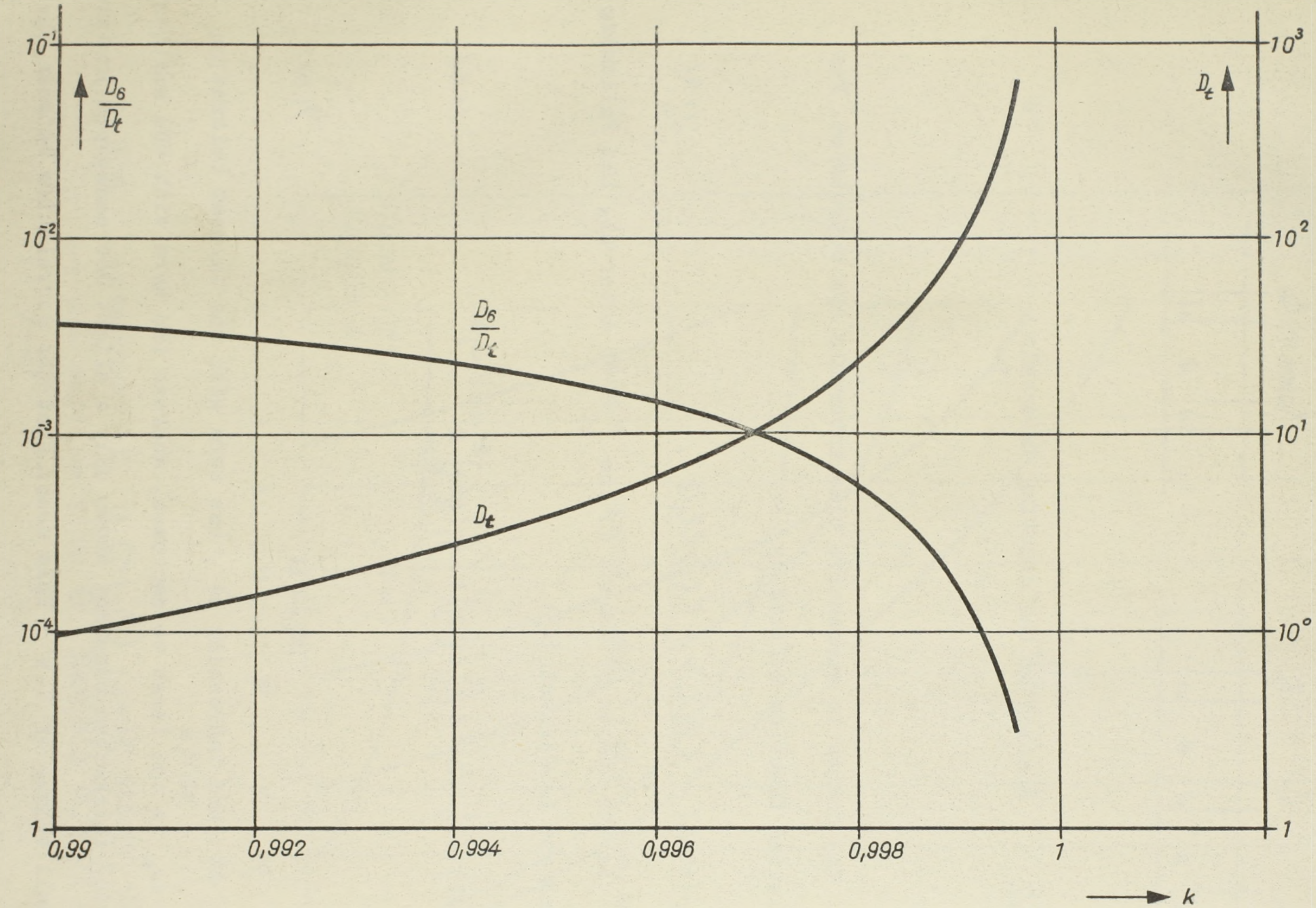


Fig.5 Variation of D_j ($j = 0,1,\ldots,6$) with k for $\tau = 5.10^{-5}$ sec



<u>Fig.6</u> Dependence of the sum $D_t=\sum_{j=0}^{6} D_j$ and the ratio D_6/D_t on the multiplication factor k for $\tau = 5.10^{-5}$ sec

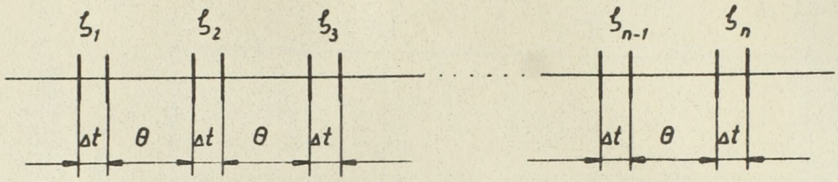


Fig.7 Spacing of the counting intervals

The counts in successive time intervals are correlated. For the correlation function defined by

$$\mathcal{K}_{ii'}(\Theta,\Delta t) = \langle \zeta_i \zeta_{i'} \rangle - \langle \zeta_i \rangle \langle \zeta_{i'} \rangle \ , \qquad /1.5/$$

where ζ_i is the number of counts in the i-th interval, the following expressions can be derived [4]:

$$\mathcal{K}_{ii'}(\Theta,\Delta t) = \frac{1}{2} N \Delta t^2 \varepsilon \sum_{j=0}^{6} \omega_j D_j \left[1 - \psi_j(\Delta t)\right]^2 e^{-|i-i'|\omega_j \Theta} \qquad /1.6/$$

$$i = i'.$$

and

$$\mathcal{K}_{ii'} = D \ . \qquad /1.7/$$

For given Δt and increasing Θ , the term with the largest relaxation constant ω_6 is the first to decrease, whereas the term with the smallest /asymptotic/ relaxation constant shows only a slight decrease, particularly for small values of $1-k$. This means that the correlation between the counts recorded in intervals Δt lying far from each other is maintained by the term due to the asymptotic relaxation constant.

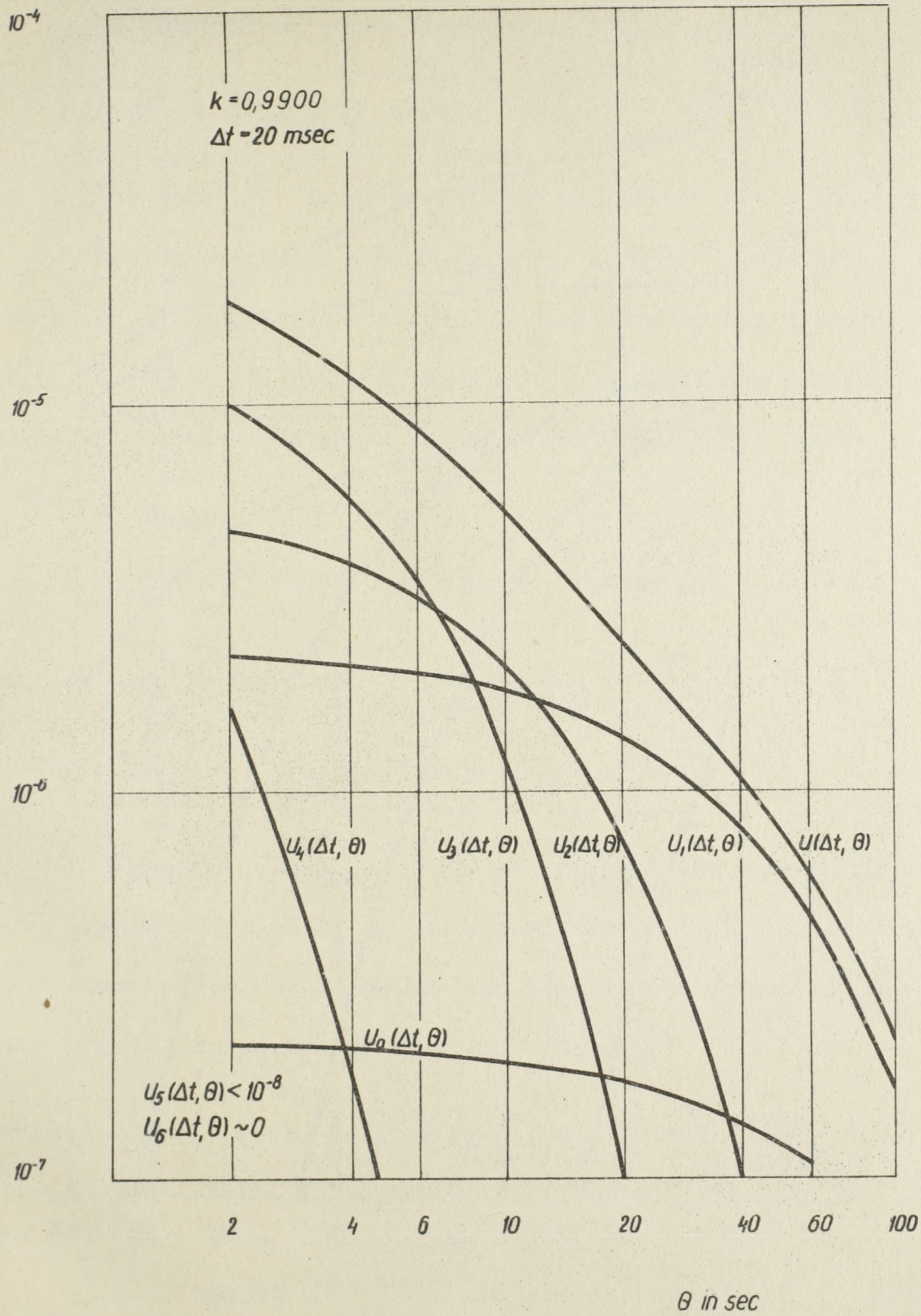


Fig.8 Dependence of $u_j(\Delta t, \Theta)$ $(j = 0,1,..6)$ and $U(\Delta t, \Theta)$ on the waiting time Θ for k = 0,9900 and $\tau = 5.10^{-5}$ sec

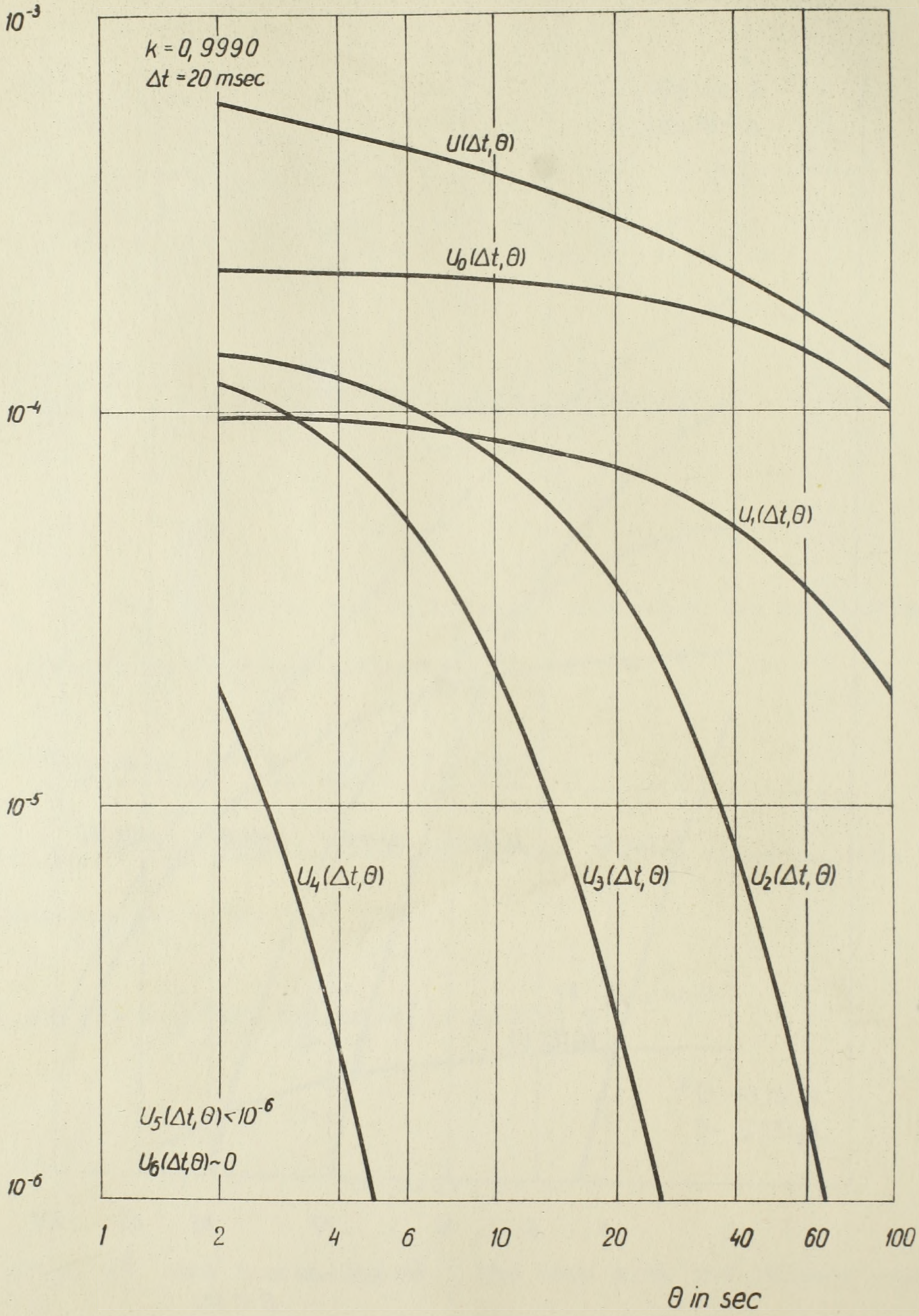


Fig.9 Dependence of $U_j(\Delta t, \Theta)$ (j 0,1,...,6) and $U(\Delta t, \Theta)$ on the waiting time Θ for k = 0,9990 and $\tau = 5.10^{-5}$ sec

In Figs. 8 and 9 the dependence of the terms

$$U_j(\Theta,\Delta t) = \Delta t^2 \omega_j D_j \left[1 - \psi_j(\Delta t)\right]^2 e^{-\omega_j \Theta} \qquad /1.8/$$

on the waiting time Θ is shown for two values of $1-k$ at a given Δt.

Let us define by $P(\Theta, \Delta t, m_1, \ldots, m_n)$ the probability that $m_1, \ldots, m_n$ counts are recorded successively by the detector in n mearuring intervals Δt, the intervals following each other after waiting times Θ. It can be shown /4/ that the probability generating function

$$G(\Theta,\Delta t, z_1, \ldots, z_n) = \sum_{m_1,\ldots,m_n} \exp(z_1 m_1 + \cdots + z_n m_n)\, P(\Theta, \Delta t, m_1, \ldots, m_n) \qquad /1.9/$$

can be rewritten in the form

$$\ln G(\Theta,\Delta t, z_1, \ldots, z_n) = i_0 \int_0^\infty \left[g(t,\Delta t,\Theta, z_1, \ldots, z_n) - 1\right] dt\,, \qquad /1.10/$$

where $g(t,\Delta t,\Theta, z_1, \ldots, z_n)$ is the probability generating function for the process initiated by an individual neutron. i_0 is the intensity of the neutron source. Applying now the Central Limit Theorem of the discrete probability variables to /1.10/, we find

$$P(\Theta, \Delta t, m_1, \ldots, m_n) \sim (2\pi)^{-\frac{n}{2}} |\mathcal{K}|^{-1} \exp\left\{-\frac{1}{2}\sum_{ii'} \varkappa_{ii'}(m_i - M)(m_{i'} - M)\right\}, \qquad /1.11/$$

where

$$M = N\Delta t \qquad /1.11'/$$

and

$$|\mathcal{K}| = \begin{vmatrix} \mathcal{K}_{11} & \mathcal{K}_{12} & \cdots & \mathcal{K}_{1n} \\ \mathcal{K}_{21} & \mathcal{K}_{22} & \cdots & \mathcal{K}_{2n} \\ \vdots & \vdots & & \vdots \\ \mathcal{K}_{n1} & \mathcal{K}_{n2} & \cdots & \mathcal{K}_{nn} \end{vmatrix} \qquad /1.12/$$

while $\varkappa_{ii'}$ is the corresponding element in the inverse of the determinant $|\mathcal{K}|$. The elements in /1.12/ are defined by the expressions /1.6/ and /1.7/. Since $\mathcal{K}_{ii} = D$ and $\mathcal{K}_{ii'} = \mathcal{K}_{|i-i'|}$ /1.12/ can be written in the form

$$|\mathcal{K}| = \begin{vmatrix} D & \mathcal{K}_1 & \mathcal{K}_2 & \cdots & \mathcal{K}_{n-1} \\ \mathcal{K}_1 & D & \mathcal{K}_1 & \cdots & \mathcal{K}_{n-2} \\ \vdots & \vdots & \vdots & & \vdots \\ \mathcal{K}_{n-1} & \mathcal{K}_{n-2} & \mathcal{K}_{n-3} & \cdots & D \end{vmatrix} . \qquad /1.13/$$

Since $D > \mathcal{K}_i$ $(i = 1, \ldots, n-1)$ the largest term in /1.13/ is obviously D^n. Consequently, the expression /1.11/ will be the more exact the better the inequality $D > 1$ is satisfied.

It is interesting to show how the correlation function $\mathcal{K}(\Theta, \Delta t)$ depends on the counting interval Δt for comparatively large waiting times. In Fig.10 the function $\mathcal{U}(\Theta, \Delta t) = \sum_{j=0}^{6} u_j(\Theta, \Delta t)$ versus Δt can be seen for $\Theta = 5$ sec at various reactivities. Using the value $\varepsilon = 10\ \text{sec}^{-1}$ for detector efficiency, the ratio $\mathcal{K}(\Theta, \Delta t)/D(\Delta t)$ has been calculated for different time intervals Δt. The results of calculation are demonstrated in Fig.11.

It is obvious that for sufficiently large Θ and small Δt it can be assumed that $\mathcal{K}_i \sim 0$. In the present case the application of the maximum likelihood method, when the distribution is determined by the function

$$\ln P(\Theta, \Delta t, m_1, \ldots, m_n) = -\frac{n}{2} \ln 2\pi D - \frac{1}{2D} \sum_{i=1}^{n} (m_i - M)^2 \qquad /1.14/$$

gives the following trivial estimates of the parameters M and D :

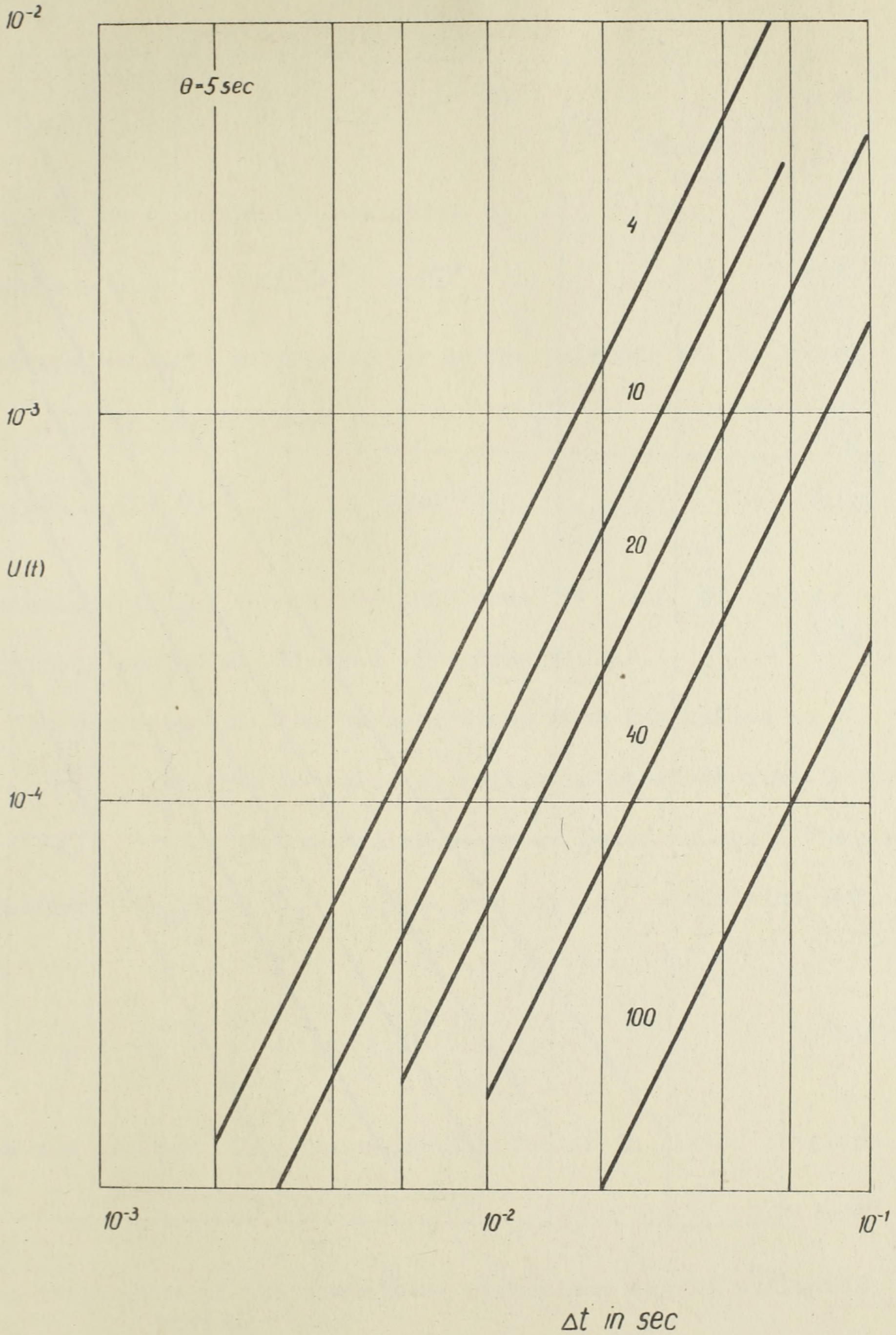


Fig.10 $U(\Delta t, \Theta)$ versus Δt for different values of $1-k$ in 10^{-4} units ($\Theta = 5$ sec)

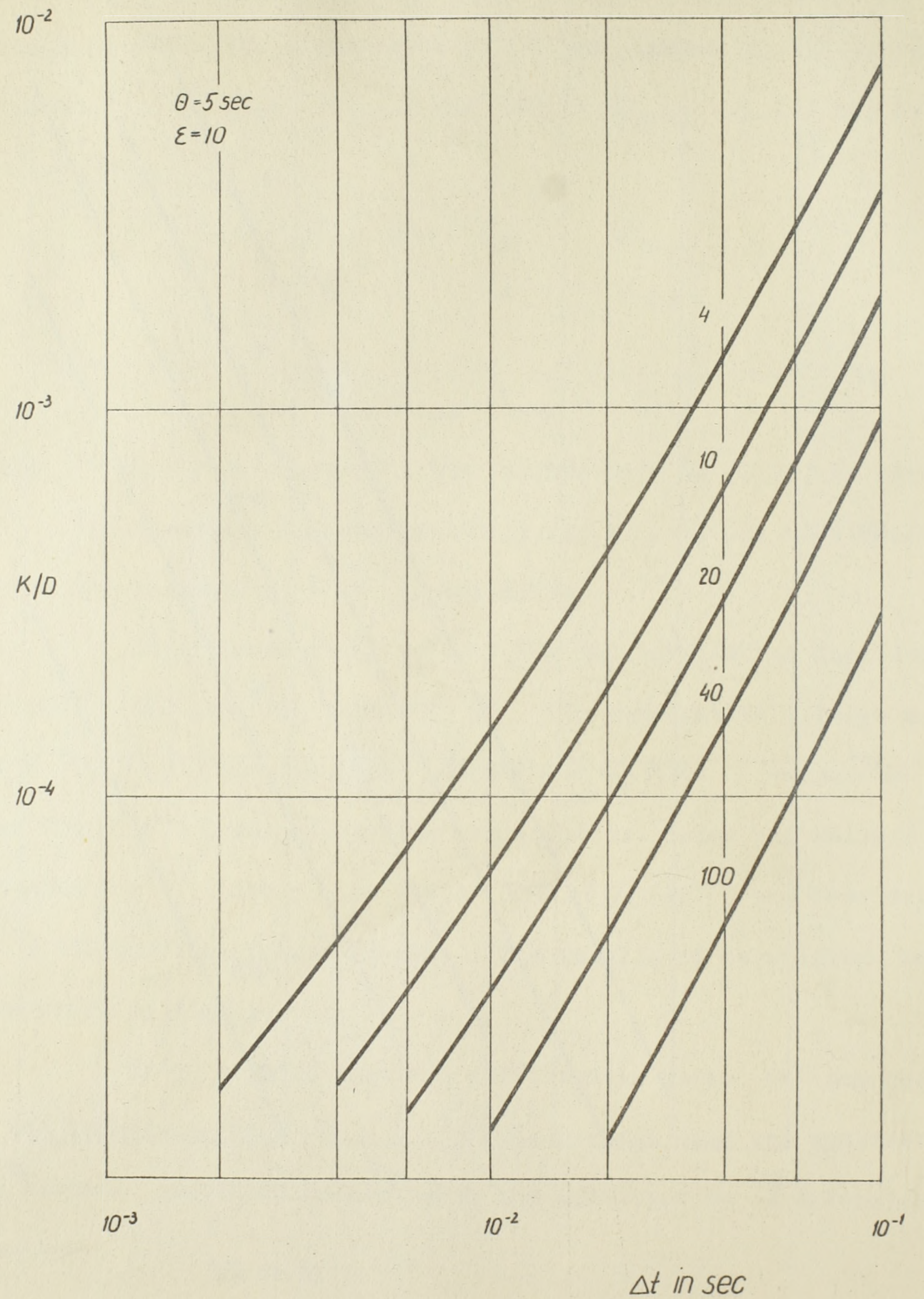


Fig.11 Dependence of the ratio $K(\Theta,\Delta t)/D(\Delta t)$ on Δt for different values of $1-k$ in 10^{-4} units ($\Theta = 5$ sec)

$$\tilde{M} = \frac{1}{n}\sum_{i=1}^{n} m_i \quad , \quad \tilde{D} = \frac{1}{n}\sum_{i=1}^{n}(m_i - \tilde{M})^2 \; . \qquad /1.15/$$

The variances of these estimates are given by

$$\langle(\delta\tilde{M})^2\rangle = D/n \quad , \quad \langle(\delta\tilde{D})^2\rangle = 2D^2/n \; . \qquad /1.16/$$

The value we are actually interested in is the variance of the ratio $\tilde{R} = \tilde{D}/\tilde{M}$. It is readily seen that

$$\langle(\delta\tilde{R})^2\rangle = \frac{2D^2}{nM^2}\left(1 + \frac{D}{2M^2}\right) + o(M^{-3/2}) \; . \qquad /1.17/$$

In the expressions /1.16/ and /1.17/ the terms M and D can be replaced by their estimates $\tilde{M}$ and $\tilde{D}$ from /1.15/.

The parameter α can be determined from the ratios $\tilde{R}_i \; (i = 1,\dots,s)$ measured at various, sufficiently short time intervals $\Delta t_i \; (i = 1,\dots,s)$ using the method of weighted least squares. The values of the parameters $\tilde{a}_1$ and $\tilde{a}_2$ $(a_1 = a$ and $a_2 = \alpha)$ minimizing the expression

$$Q = -\frac{1}{2}\sum_{i=1}^{s} w_i\left[\tilde{R}_i - 1 - a_1\psi_i(a_2)\right]^2 \qquad /1.18/$$

can be obtained by iteration. The weighting factor in /1.18/ is given by

$$w_i = \langle(\delta\tilde{R})^2\rangle^{-1} \; . \qquad /1.18'/$$

The r-th iteratives of the minimizing parameters can be evaluated from the equation

$$\tilde{a}_\mu(r) = \tilde{a}_\mu(r-1) + \sum_{\nu=1}^{2} F_\nu\left[\tilde{a}_1(r-1), \tilde{a}_2(r-1)\right] R_{\nu\mu}\left[\tilde{a}_1(r-1), \tilde{a}_2(r-1)\right] \; , \qquad /1.19/$$

$$(\mu = 1, 2)$$

where

$$F_\nu = \frac{\partial Q}{\partial a_\nu} \qquad (\nu = 1,2) \tag{1.20}$$

and $R_{\nu\mu}$ is the corresponding element of the inverse of the matrix $\underline{S}$ with the elements

$$S_{\nu\mu} = \left\langle \frac{\partial F_\nu}{\partial a_\mu} \right\rangle \quad . \tag{1.21}$$

$$(\nu,\mu = 1,2)$$

To carry out the above calculations first of all, the initial values $\tilde{a}_1(0)$ and $\tilde{a}_2(0)$ have to be estimated. Starting the iteration procedure with the initial values $\tilde{a}_1(0)$ and $\tilde{a}_2(0)$, a few steps are, in general, sufficient to yield fairly accurate estimates of the parameters a_1 and a_2. The variance of the estimated values is given by

$$\langle(\delta\tilde{a}_\mu)^2\rangle = -R_{\mu\mu} \quad , \tag{1.22}$$

$$(\mu = 1,2)$$

where $R_{\mu\mu}$ is conveniently calculated from the last iteratives of the parameters a_μ $(\mu = 1,2)$.

The ratio β^{eff}/ℓ can be given with relatively small error if the number of different $\tilde{\alpha}$ -s is large and their variances are small enough. The variance $\langle(\delta\tilde{\alpha})^2\rangle$ can be reduced considerably by increasing the number of different time intervals used in the measuring of $\tilde{\alpha}$ and decreasing the variance of the $\tilde{R}$. We have seen that the variance $\langle(\delta\tilde{R})^2\rangle$ is proportional to $(D/M)^2$, therefore the number of measurements n for a given time interval Δt must be large, particularly if

$D/M \gg 1$. In practice, it is necessary to record a few thousand counts for each time interval.

Correlation method

The correlation between the counts recorded in consecutive time intervals diminishes with the increase of time between the intervals. The first to decrease is the term with the largest relaxation constant, and gradually all the other terms become relatively small too. This fact will be exploited for the experimental determination of the largest relaxation constant.

First of all, we determine the conditional probability of a count in a time interval Δt_2 following a count in an earlier time interval Δt_1 . Let $W(t_2-t_1, \Delta t_2 | \Delta t_1)$ denote this conditional probability. The spacing of the time intervals is illustrated in Fig.12. The expression for $W(t_2-t_1, \Delta t_2 | \Delta t_1)$ has already been derived by several authors [5].

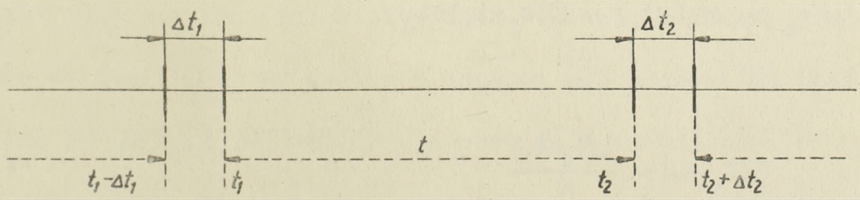


Fig.12 Spacing of the time intervals

We should like here to give a quite general expression for $W(t_2-t_1,\Delta t_2|\Delta t_1)$ taking into account also the effect of delayed neutrons. If $W(t_2-t_1,\Delta t_2,\Delta t_1)$ is the probability of a count in each of the time intervals Δt_2 and Δt_1, we may write

$$W(t_2-t_1,\Delta t_2|\Delta t_1) = \frac{W(t_2-t_1,\Delta t_2,\Delta t_1)}{W(\Delta t_1)} , \qquad /2.1/$$

where $W(\Delta t_1)$ represents the probability that at least one count is recorded in the interval Δt_1. Assuming the intervals Δt_2 and Δt_1 to be sufficiently small, we find

$$W(t_2-t_1,\Delta t_2,\Delta t_1) = w(t_2-t_1)\Delta t_2\,\Delta t_1 + o(\Delta t_2\,\Delta t_1) , \qquad /2.2/$$

unless Δt_2 and Δt_1 have a common point. If $t_2 = t_1-\Delta t_1$ and $\Delta t_1=\Delta t_2=\Delta t$ we have

$$W(t_2-t_1,\Delta t_2,\Delta t_1) \longrightarrow W(\Delta t) . \qquad /2.3/$$

Similar to the expression /2.2/ we obtain

$$W(\Delta t_1) = N\Delta t_1 + o(\Delta t_1) \qquad /2.4/$$

and thus

$$W(t_2-t_1,\Delta t_2|\Delta t_1) \sim C(t_2-t_1)\Delta t_2 , \qquad /2.5/$$

where

$$C(t_2-t_1) \sim \frac{w(t_2-t_1)}{N} . \qquad /2.6/$$

For the more exact definition of N and $w(t_2-t_1)$ see Appendix I.

The probability $w(t_2-t_1)\,dt_2\,dt_1$ can be easily calculated if the distribution function of the number of neutrons in the reactor is known. Let be $P(m', m'_1, \dots, m'_6)$ the probability that precisely m' neutrons and $m'_1, \dots, m'_6$ fission-product nuclei capable of giving off delayed neutrons are present in the steadily operating reactor at time $t_1 - dt_1$. It is obvious that the probability of a count in the time interval $(t_1 - dt_1, t_1)$ is $\varepsilon m' dt_1 + o(dt_1)$, where ε is the detector efficiency. Let us denote by $P(t_2 - t_1, m | m'-1, m'_1, \dots, m'_6)$ the conditional probability that m neutrons are present in the reactor at time $t_2 > t_1$ if $m'-1$ neutrons and $m'_1, \dots, m'_6$ fission-product nuclei were present at time t_1 . It is easy to see that

$$w(t_2 - t_1) dt_2 dt_1 =$$

$$= \varepsilon^2 dt_2 dt_1 \sum_{m', m'_1, \dots, m'_6} m' P(m', m'_1, \dots, m'_6) \sum_{m=0}^{\infty} m P(t_2 - t_1, m | m'-1, m'_1, \dots, m'_6) \qquad /2.7/$$

Now we have to determine the conditional probability $P(t_2 - t_1, m | m'-1, m'_1, \dots, m'_6)$. Since each of the $m'-1$ neutrons may start a chain reaction independently of the others we have

$$p(t_2 - t_1, m_r | m'-1) = \sum_{m_1 + \dots + m_{m'-1} = m_r} \prod_{i=1}^{m'-1} p(t_2 - t_1, m_i) , \qquad /2.8/$$

where $p(t_2 - t_1, m_i)$ is the probability that m_i neutrons are present in the reactor at time t_2 if a single neutron was present at time $t_1 < t_2$. /The meaning of $p(t_2 - t_1, m_r | m'-1)$ is obvious./ In addition to the chain reactions started by the neutrons present at time t_1 , chain reactions can be initiated also by the neutrons emitted in the interval (t_1, t_2) by the fission-product nuclei. Denote by $P(t_2 - t_1, m_f | m'_1, \dots, m'_6)$ the probability that m_f neutrons are present in the reactor at time t_2 , if $m'_1, \dots m'_6$

fissíon-product nuclei capable of giving off delayed neutrons were present at time t_1 . Finally, we have to consider the chain reactions initiated by the source neutrons. Denote by $P(t_2-t_1, m_s)$ the probability that m_s neutrons are present in the reactor at time $t_2 > t_1$ provided that the chain reaction may be started only by source neutrons in the time interval (t_1, t_2) . It follows from the above that we may write

$$P(t_2-t_1, m| m'-1, m'_1, \dots, m'_6) =$$

$$= \sum_{m_r+m_f+m_s=m} p(t_2-t_1, m_r| m'-1) P(t_2-t_1, m_f| m'_1, \dots, m'_6) P(t_2-t_1, m_s). \qquad /2.9/$$

Introducing now the probability generating function

$$G(t_2-t_1, z| m'-1, m'_1, \dots, m'_6) = \sum_{m=0}^{\infty} e^{mz} P(t_2-t_1, m| m'-1, m'_1, \dots, m'_6) \qquad /2.10/$$

and putting $t_2-t_1 = t > 0$

$$G(t_2-t_1, z| m'-1, m'_1, \dots, m'_6) = [g(t,z)]^{m'-1} G(t,z) \prod_{i=1}^{6} [g_i(t,z)]^{m'_i} , \qquad /2.11/$$

where

$$G(t,z) = \exp\left\{-i_0 \int_0^t [1-g(t',z)]dt'\right\}, \qquad g_i(t,z) = e^{-\lambda_i t} + \lambda_i \int_0^t e^{-\lambda_i t'} g(t-t',z)dt' . \qquad /2.12/$$

/For the derivation of /2.12/ see the papers [4] ./

From the equation /2.7/ we have

$$w(t) = \varepsilon^2 \sum_{m', m'_1, \dots, m'_6} m' P(m', m'_1, \dots, m'_6) M(t| m'-1, m'_1, \dots, m'_6) , \qquad /2.13/$$

where

$$M(t| m'-1, m'_1, \dots, m'_6) = \left(\frac{\partial G(t, z| m'-1, m'_1, \dots, m'_6)}{\partial z}\right)_{z=0} . \qquad /2.13'/$$

Using the following notations:

$$m(t) = \left(\frac{\partial g(t,z)}{\partial z}\right)_{z=0} \quad \text{and} \quad M(t) = \left(\frac{\partial G(t,z)}{\partial z}\right)_{z=0} \qquad /2.14/$$

we write

$$M(t|\, m'-1, m_1', \ldots, m_6') =$$

$$= M(t) + (m'-1)\, m(t) + \sum_{i=1}^{6} m_i' \lambda_i \int_0^t e^{-\lambda_i t'}\, m(t-t')\, dt' \, . \qquad /2.15/$$

Finally, we have

$$w(t) = \varepsilon^2 \Big\{ M_1 M(t) + (M_2 - M_1)\, m(t) + \sum_{i=1}^{6} Q_i \lambda_i \int_0^t e^{-\lambda_i t'} m(t-t')\, dt' \, , \qquad /2.16/$$

where

$$M_k = \langle (m')^k \rangle \quad \text{and} \quad Q_i = \langle m_i' m' \rangle . \qquad /2.17/$$

Introducing the expressions for $m(t)$ and $M(t)$ given in Appendix II into /2.16/ and taking into account that $N = \varepsilon M_1$ we find:

$$C(t) = i_0 \varepsilon \sum_{j=0}^{6} \frac{c_j}{\omega_j} + \varepsilon \sum_{j=0}^{6} \left[\frac{M_2 - M_1}{M_1} + \frac{F_j}{M_1} - \frac{i_0}{\omega_j} \right] c_j e^{-\omega_j t} \, , \qquad /2.18/$$

where

$$c_j = \ell \omega_j \left[\frac{\beta^{eff}}{\beta} \sum_{i=1}^{6} \frac{\beta_i}{\left(1 - \frac{\lambda_i}{\omega_j}\right)^2} - \frac{k-1}{k} \right]^{-1} ,$$

$$F_j = \sum_{i=1}^{6} \frac{Q_i \lambda_i}{\lambda_i - \omega_j} \, . \qquad /2.19/$$

It can be easily shown, however, that the following relationship holds for a steadily operating subcritical reactor containing a neutron source:

$$c_j \left[\frac{M_2 - M_1}{M_1} + \frac{F_j}{M_1} - \frac{i_0}{\omega_j} \right] = \frac{1}{2} \omega_j D_j \qquad /2.20/$$

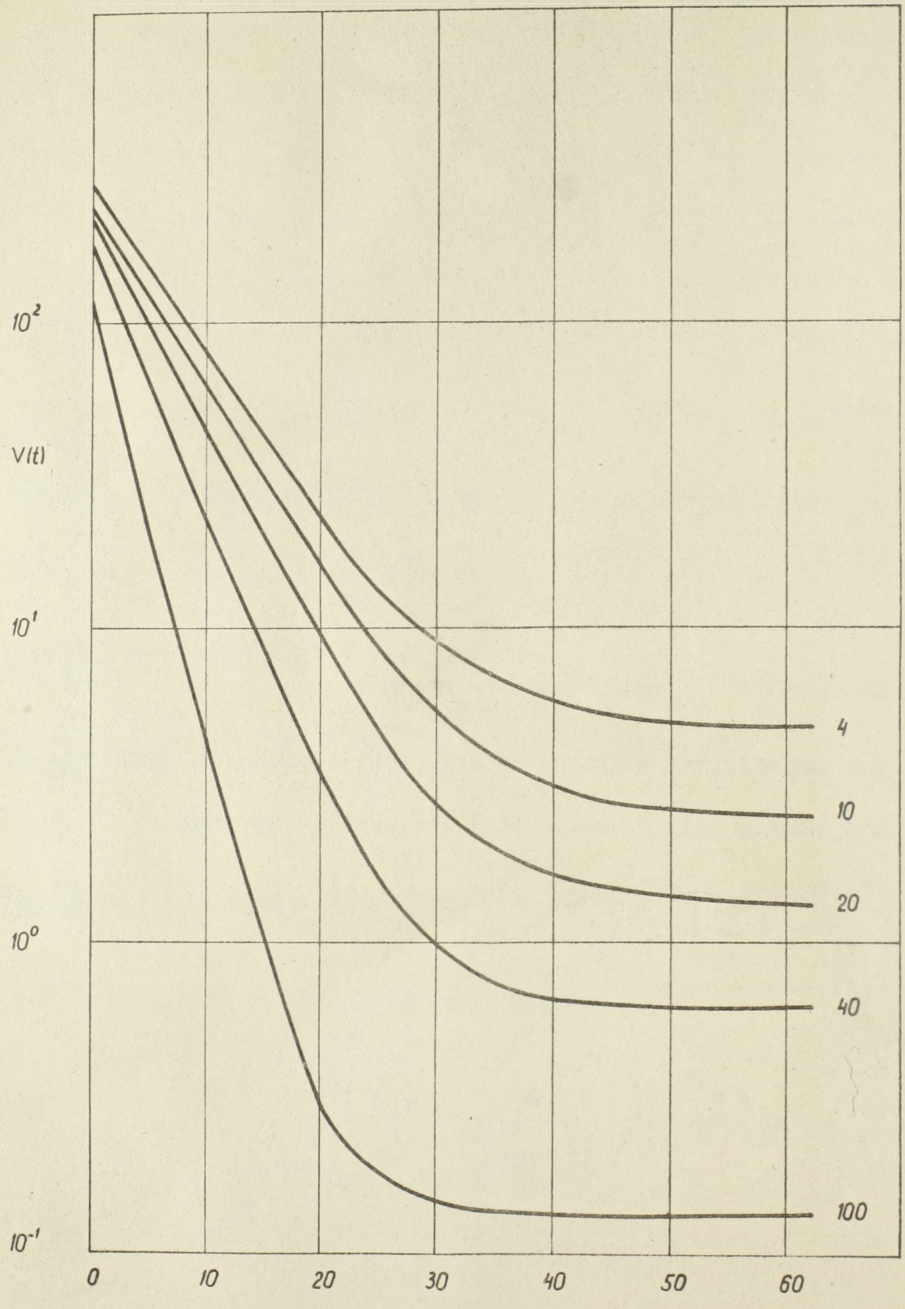


Fig.13 Time dependence of $V(t)$ for different values of $1-k$ in 10^{-4} units

and thus

$$C(t) = N + \frac{1}{2}\varepsilon V(t) = N + \frac{1}{2}\varepsilon \sum_{j=0}^{6} V_j(t) \ , \qquad /2.21/$$

where

$$N = i_0 \frac{\varepsilon \ell k}{1-k} \qquad /2.22/$$

and

$$V_j(t) = \omega_j D_j e^{-\omega_j t} \ . \qquad /2.23/$$

In Fig.13 we have shown the time dependence of $V(t)$ at various reactivities. The components $V_j(t)$ $(j=0,\ldots,6)$ are shown in Fig.14, resp. 15 at two reactivities. It is obvious that the experimental conditions are the most favourable when leading to so fast a decrease in the term $V_6(t)$ that the other terms have no time to change appreciably. Of course, it is rather difficult to achieve this condition and the inaccuracy in a number of experimental results is probably due to the fact that the components decreasing slower than $V_6(t)$ have not been taken properly into account when evaluating the experimental data.

Assuming now that ideal experimental conditions prevail, that is for each value of $t < t_k$ we have $e^{-\omega_j t} \sim 1$ $(j = 0, \ldots, 5)$, then

$$C(t) \sim a_1 + a_2 e^{-a_3 t} \ , \qquad /2.24/$$

where

$$a_1 = N + \frac{1}{2}\varepsilon \sum_{j=0}^{5} \omega_j D_j \ ,$$

$$a_2 = \frac{1}{2}\varepsilon \omega_6 D_6 \qquad \text{and} \qquad a_3 = \omega_6$$

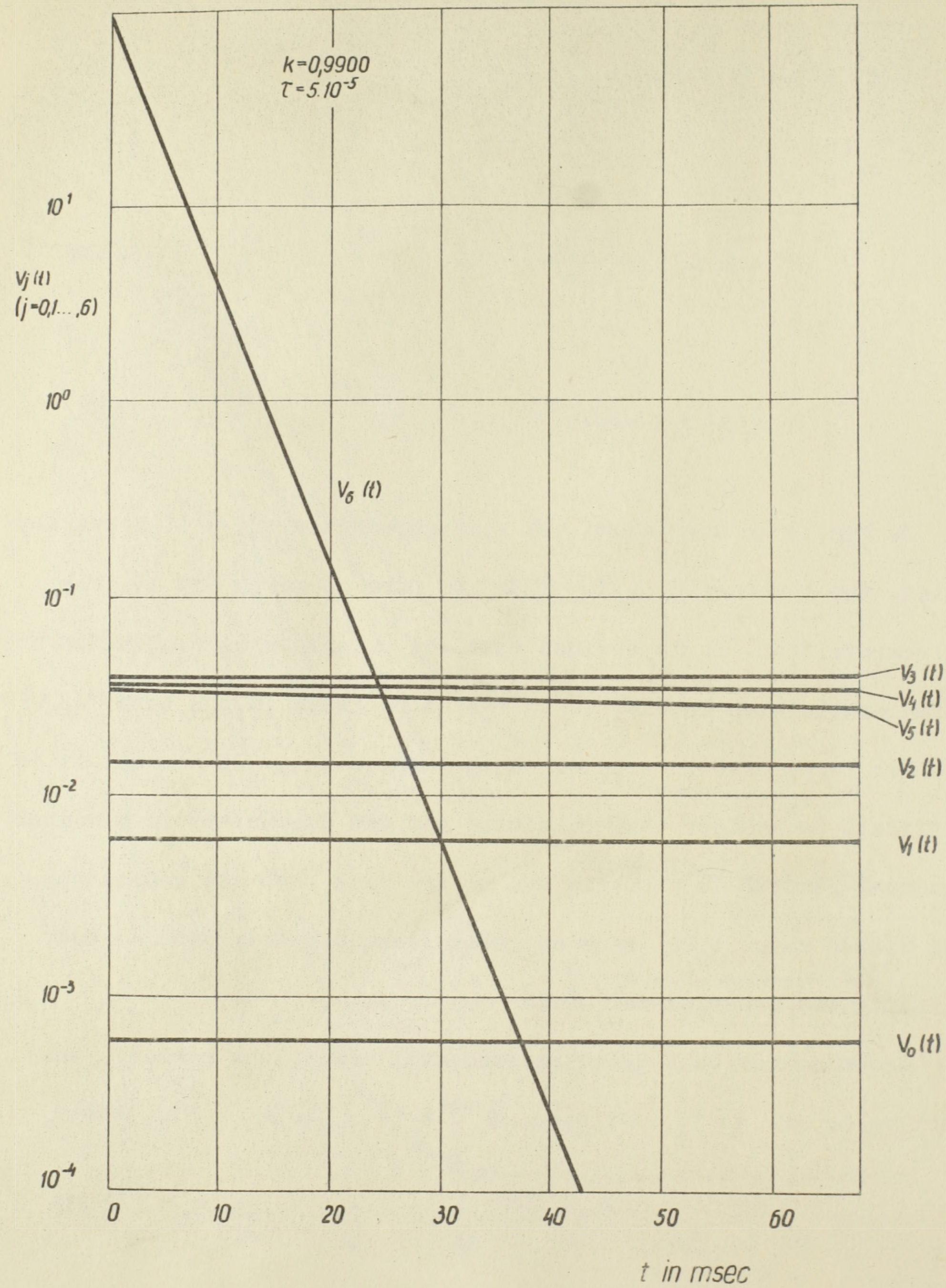
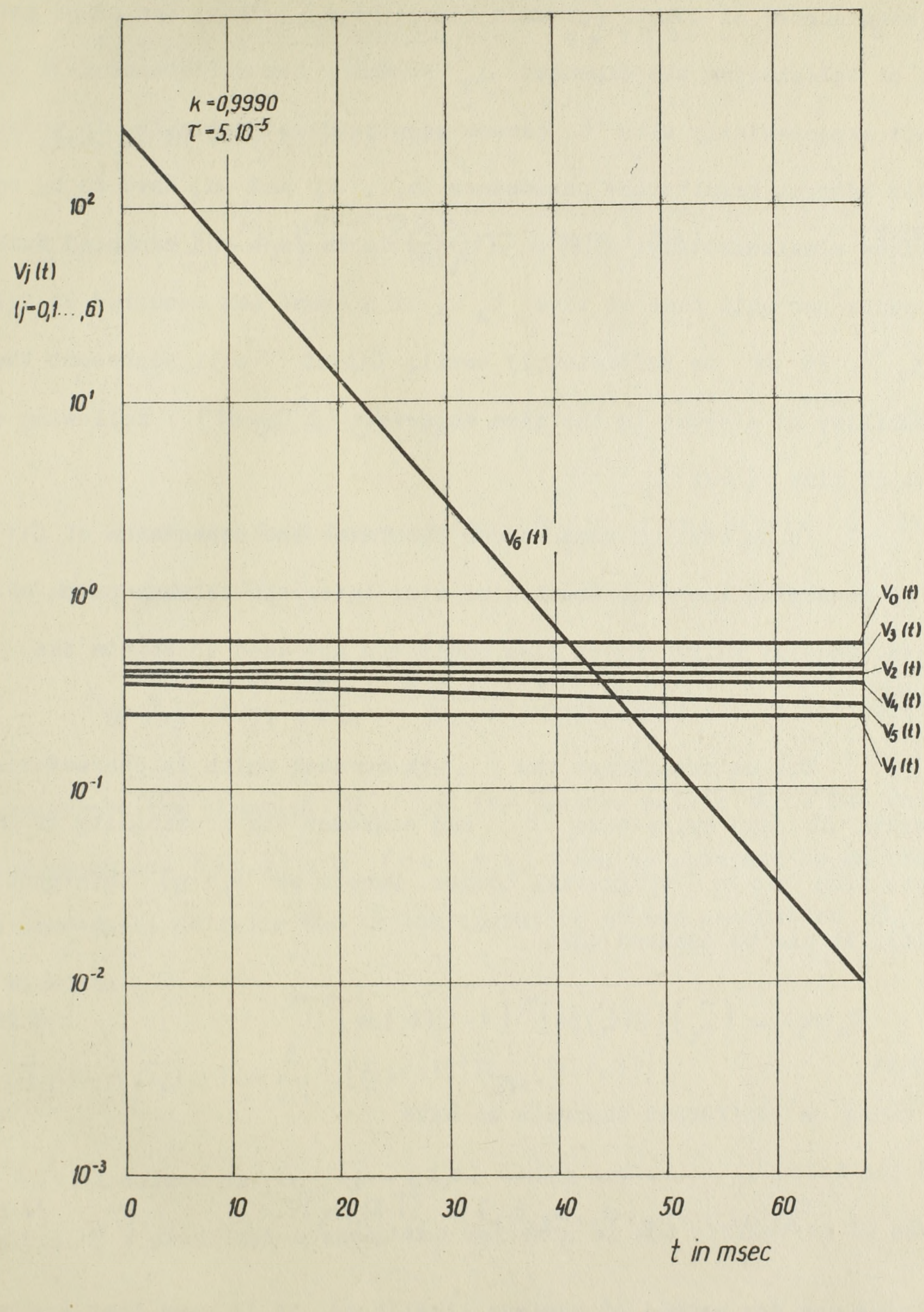


Fig.14 Dependence of $V_j(t)$ $(j=0,1,\dots,6)$ on the time t for $k = 0{,}9900$



<u>Fig.15</u> Dependence of $V_j(t)$ $(j=0,1,\ldots,0)$ on the time t for k = 0,9990

It is apparent from the above equations that a_1 is not identical with the mean number of counts per unit time $(N \neq a_1)$. Thus, the usual procedure of calculating the exponent a_3 assuming the difference $C-N$ to decrease exponentially with t cannot be considered reliable. In order to obtain correct results the parameters a_1, a_2 and a_3 have to be determined simultaneously. $C(t) = C(t_2-t_1)$ is in fact the expected number of counts per unit time at time t_2, if a count has occurred at time $t_1 < t_2$. If dt is sufficiently small, $C(t)dt$ will represent the probability of a count in the time interval $(t, t+dt)$ following a count at time $t=0$.

It is obvious that we can determine the dependence of $C(t)$ on t by measuring the distribution of time intervals between pairs of counts. Usually multichannel time analysers are used to obtain the distribution of the time intervals.

Let us now choose the i-th channel which is characterized by width Δt and delay time t_i and consider the probability of n_i counts from $n > n_i$ triggering counts. Denote by $P_n(n_i)$ this probability. It can be assumed that

$$P_n(n_i) = \binom{n}{n_i}\left[C(t_i)\Delta t\right]^{n_i}\left[1-C(t_i)\Delta t\right]^{n-n_i} \qquad /2.25/$$

and using s different channels we have

$$P_n(n_1, \dots, n_s; a_1, a_2, a_3) = \prod_{i=1}^{s} P_n(n_i) \, . \qquad /2.26/$$

Applying now the method of maximum likelihood, it is seen that the best estimation of the parameters a_1, a_2, a_3 can be obtained by evaluating the quantities $\tilde{a}_1, \tilde{a}_2, \tilde{a}_3$ for which the expressions

$$F_\nu(\underline{n},\underline{a}) = \frac{\partial \ln P_n(\underline{n},\underline{a})}{\partial a_\nu} \qquad (\nu = 1,2,3) \qquad /2.27/$$

become zero. It is easy to show that

$$F_\nu(\underline{n},\underline{a}) = \sum_{i=1}^{s} \frac{n_i - nB_i}{1-B_i} \frac{\partial \ln B_i}{\partial a_\nu} \qquad (\nu = 1,2,3) \ , \qquad /2.28/$$

where

$$B_i = C(t_i)\Delta t \ . \qquad /2.29/$$

Let us now find the elements $S_{\nu\mu}$ of the matrix $\underline{S}$ with help of the relationship

$$S_{\nu\mu} = -n \sum_{i=1}^{s} \frac{B_i}{1-B_i} \frac{\partial \ln B_i}{\partial a_\nu} \frac{\partial \ln B_i}{\partial a_\mu} \qquad /2.30/$$

and construct the elements $R_{\nu\mu}$ of its inverse matrix $\underline{R}$. The roots of the equations $F_\nu(\underline{n},\underline{\tilde{a}}) = 0$ $(\nu = 1,2,3)$ can be obtained by an iteration procedure, defining the r-th iterative of the estimation $\tilde{a}_\mu$ by the equation

$$\tilde{a}_\mu(r) = \tilde{a}_\mu(r-1) + \sum_{\nu=1}^{3} F_\nu[\underline{\tilde{a}}(r-1)]\, R_{\nu\mu}[\underline{\tilde{a}}(r-1)] \ . \qquad /2.31/$$

The initial values of $\tilde{a}_\mu$ $(\mu = 1,2,3)$ which are to be iterated can be obtained by a graphical method. The variance of the estimation is given by

$$\langle(\delta\tilde{a}_\mu)^2\rangle = -R_{\mu\mu} \qquad (\mu = 1,2,3) \ , \qquad /2.32/$$

where $R_{\mu\mu}$ is calculated from the last iteratives of the parameters a_1, a_2, a_3.

Although we are actually interested in the parameter $a_3 = \omega_6 \sim \alpha$ only, this cannot be evaluated independently of the others. Nevetheless a choice of the experimental conditions is feasible, where the variance $\langle(\delta\tilde{\alpha})^2\rangle$ is kept at a minimum and in this case the parameter of interest to us can be estimated more accurately at the expense of the other parameters involved.

In several experiments reported in the literature [6], [7] the authors measured the distribution of the time intervals between two successive counts. Denote by $C_1(t)$ the probability density function of these intervals. The evaluation of the measured data was done, unfortunately, using $C(t)$ as given in /2.24/, although it is obvious that $C(t)$ corresponds to the distribution of time intervals between any two counts.

We show presently the relation between the functions $C_1(t)$ and $C(t)$. If $C_k(t)dt$ is the probability that the k-th count is recorded in the time interval $(t, t+dt)$ following the triggering count at time $t = 0$ then

$$C(t) = \sum_{k=1}^{\infty} C_k(t). \qquad /2.33/$$

Thus in principle $C(t)$ cannot be used instead of $C_1(t)$. It can be shown, however, /see Appendix III/ that for detectors of rather low efficiency /an undesirable condition for other reasons/ we may find

$$C_1(t) \sim C(t) + o(\varepsilon) , \qquad /2.34/$$

where $o(\varepsilon)$ represents higher than first order terms in ε .

The experiments referred to above have been possibly performed under conditions where $C_1(t)$ did not differ appreciably from $C(t)$, although the results obtained by Ingram et al. [7] seem to indicate that these conditions have not been met.

Zero-probability method

A remarkable method has been suggested recently by Moguilner [8] for the determination of the relaxation constant $\omega_6 \sim \alpha$. The probability of obtaining no count in a suitably chosen time interval Δt is determined experimentally and from the probabilities measured for different values of Δt , α is then computed.

Moguilner presumed that the probability of recording m counts during the time Δt may be expressed by the negative binomial distribution function. Denoting this probability by $P_M(\Delta t, m)$, he obtained for the generating function

$$G_M(\Delta t, z) = \sum_{m=0}^{\infty} e^{mz} P_M(\Delta t, m) \qquad /3.1/$$

the following expression:

$$G_M(\Delta t, z) = \left[1 + \varepsilon (1 - e^z) \phi_6(\Delta t) \right]^{-\frac{N\Delta t}{\varepsilon \phi_6(\Delta t)}} \qquad /3.2/$$

from which the logarithm of the zero probability can be written in the form:

$$\ln P_M(\Delta t,0) = \ln G_M(\Delta t,-\infty) = -\frac{N\Delta t}{\varepsilon\phi_6(\Delta t)}\ln\left[1+\varepsilon\phi_6(\Delta t)\right] , \qquad /3.3/$$

where

$$\phi_6(\Delta t) = D_6\psi_6(\Delta t) .$$

From the values of $P_M(\Delta t,0)$ determined experimentally $\phi_6(\Delta t)$ and thus also α may be computed.

The formula /3.2/ derived by Moguilner needs, however, a proper theoretical basis. It seemed therefore reasonable to derive a reliable expression for the zero probability on the basis of the method elaborated previously by us [4]. According to the exact theory [4] the zero probability may be computed from the relation

$$P(\Delta t,0) = \exp\left\{-i_0\int_0^\infty \pi(\Delta t,t)dt\right\} , \qquad /3.4/$$

where $\pi(\Delta t,t)$ is the probability of recording no counts in the time interval $(t-\Delta t,t)$ provided that there was only one neutron in the reactor at time $t=0$. We have shown earlier that $\pi(\Delta t,t)$ satisfies the equation

$$\frac{d\pi(\Delta t,t)}{dt} = \varepsilon\Delta(\Delta t-t)+\alpha_f-(\alpha_f+\alpha_c)\pi(\Delta t,t)-\alpha_f q\left[\vec{x}(\Delta t,t)\right] , \qquad /3.5/$$

where $q(\vec{x})$ is the generating function characterizing the distribution of the number of fission neutrons, α_f and α_c are the reciprocal quantities of fission and absorption life times of the neutrons, respectively. The components of the vector $\vec{x}=(x_0,\dots,x_6)$ are given by the relations

$$x_0 = \ln\left[1 - \pi(\Delta t, t)\right] ,$$

$$x_i = \ln\left[1 - \lambda_i \int_0^t e^{-\lambda_i t'} \pi(\Delta t, t - t')\,dt'\right] . \qquad /3.6/$$

$$(i = 1, \ldots, 6)$$

The initial condition corresponding to /3.5/ is given by

$$\lim_{t \to 0} \pi(\Delta t, t) = 0 . \qquad /3.7/$$

Disregarding the effects of delayed neutrons and considering solutions only for which $\pi(\Delta t, t) << 1$ it is sufficient to investigate the approximate equation

$$\frac{d\pi(\Delta t, t)}{dt} = \varepsilon \Delta(\Delta t - t) - \alpha \pi(\Delta t, t) - \frac{1}{2} \alpha_f (\nu_{00} - \nu_0)\left[\pi(\Delta t, t)\right]^2 \qquad /3.8/$$

only. We get

$$\pi(\Delta t, t) = \begin{cases} \dfrac{2\varepsilon}{\alpha} \dfrac{1 - e^{-\alpha\gamma\Delta t}}{\gamma + 1 + (\gamma - 1) e^{-\alpha\gamma\Delta t}} , & \text{if} \quad t \leq \Delta t \\[2ex] \dfrac{c e^{-\alpha t}}{1 - \frac{1}{2} c \alpha D_6 \, e^{-\alpha t}} , & \text{if} \quad t > \Delta t , \end{cases} \qquad /3.9/$$

where

$$\gamma = \sqrt{1 + 2\varepsilon D_6} \qquad /3.10/$$

and

$$C = \frac{2\varepsilon}{\alpha} \frac{e^{\alpha\Delta t}(1 - e^{-\alpha\gamma\Delta t})}{\gamma + 1 + \varepsilon D_6 + (\gamma - 1 - \varepsilon D_6) e^{-\alpha\gamma\Delta t}} . \qquad /3.11/$$

From /3.4/, /3.9/, /3.10/ and /3.11/ we have

$$\ln P(\Delta t, o) = - N\Delta t \frac{2}{\gamma+1}\left[1 + \frac{2}{(\gamma-1)\alpha\Delta t}\ln\frac{(\gamma+1)^2-(\gamma-1)^2 e^{-\alpha\gamma\Delta t}}{4\gamma}\right]. \qquad /3.12/$$

It can be shown easily that the first two terms of the expression /3.12/ expanded in a power series in εD_6 are the same as the corresponding terms for $\ln P_M(\Delta t, o)$. Naturally, the expansion of /3.12/ into a power series in εD_6 may be justified only when $\varepsilon D_6 < 1$. This condition, however, means that the variance of the counts is hardly different from that of a Poisson distribution. The determination of α , on the other hand, is the easier the greater the deviation from a Poisson distribution, that is the better the condition $\varepsilon D_6 > 1$ is fulfilled. So we see that $\ln P(\Delta t, o)$ cannot be replaced by $\ln P_M(\Delta t, o)$.

In Fig.16 the probabilities $P_M(\Delta t, o)$ and $P(\Delta t, o)$ can be seen as functions of Δt for different reactivities.

For the determination of α the following procedure has to be resorted to. One determines experimentally the number of time intervals Δt in which no counts have been recorded. When the waiting time between the intervals is long enough, the events in two successive intervals of length Δt are almost independent. Therefore we may assume that

$$P(n,k,\Delta t) = \binom{n}{k}\left[P(\Delta t, o)\right]^k \left[1 - P(\Delta t, o)\right]^{n-k} \qquad /3.13/$$

is the probability of obtaining no counts in k cases out of a total number of $n > k$ measurements.

For the evaluation of measurements carried out with time intervals of different lengths Δt_i $(i = 1, \ldots, s)$ the following maximum

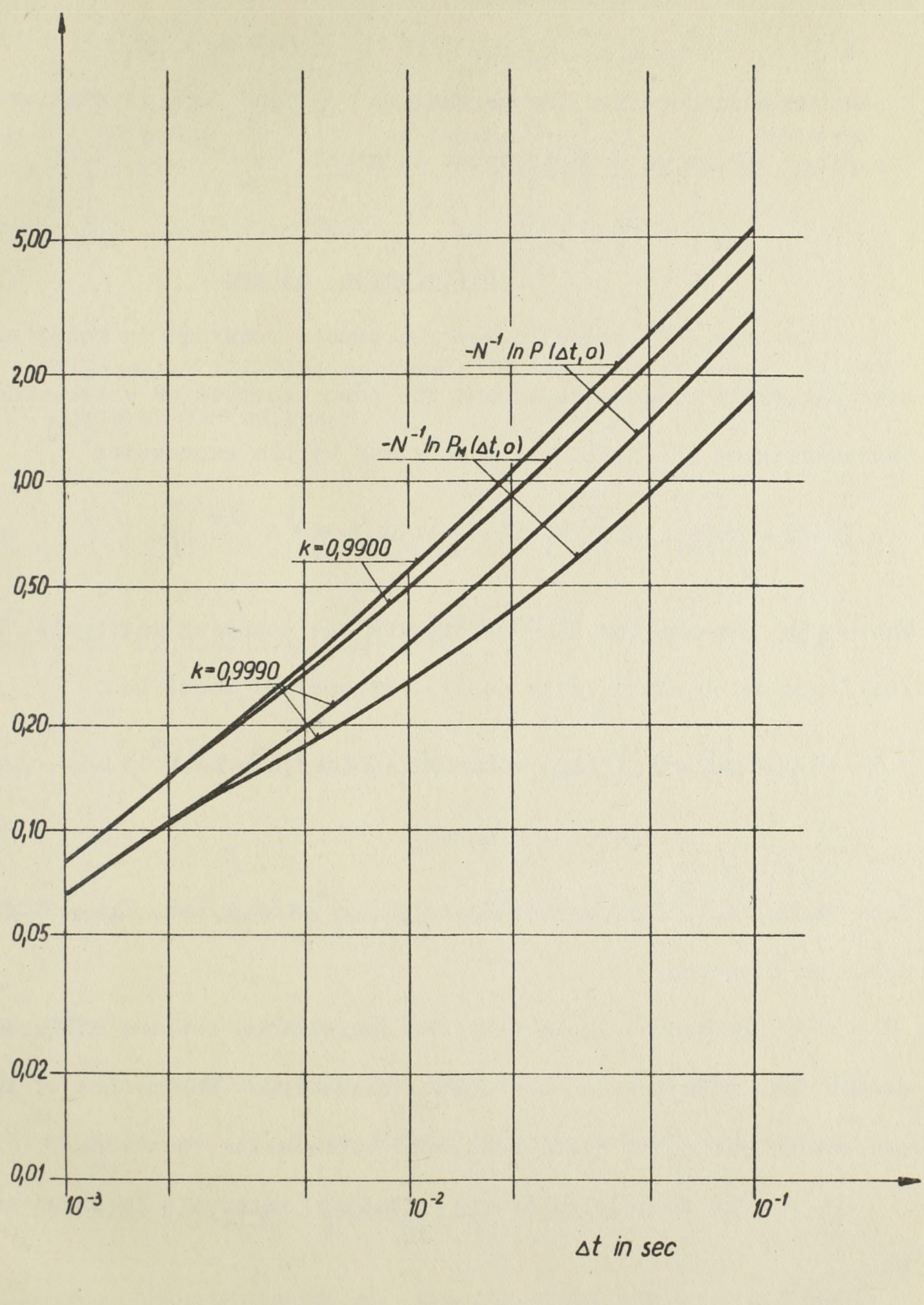


Fig.16 $N^{-1} \ln P_M(\Delta t, 0)$ and $N^{-1} \ln P(\Delta t, 0)$ versus Δt for two values of k

likelihood function can be used:

$$\ln P(n_1,\dots,n_s,k_1,\dots,k_s,\Delta t_1,\dots,\Delta t_s) = \sum_{i=1}^{s} \ln P(n_i,k_i,\Delta t_i). \qquad /3.14/$$

Since the procedure for the parameters γ and α is similar to that described before we do not repeat it here.

Differential method

The difference in neutron counts recorded in equal successive time intervals fluctuates around the zero expectation value. The variance characterizing the fluctuation is given by the expression

$$D_\delta(\Theta,\Delta t) = 2N\Delta t\left\{1+\varepsilon\sum_{j=0}^{6} D_j\left[\psi_j - \frac{1}{2}\omega_j\Delta t(1-\psi_j)^2 e^{-\omega_j\Theta}\right]\right\}, \qquad /4.1/$$

where Θ denotes the time separating the counting intervals. The result /4.1/ can be obtained quite easily. Appendix I leads to

$$D_\delta(\Theta,\Delta t) = \left\langle\left[\zeta(t_1,t_1+\Delta t)-\zeta(t_1+\Theta+\Delta t,t_1+\Theta+2\Delta t)\right]^2\right\rangle =$$

$$= 2D(\Delta t) - 2\mathcal{K}(\Theta,\Delta t) \qquad /4.2/$$

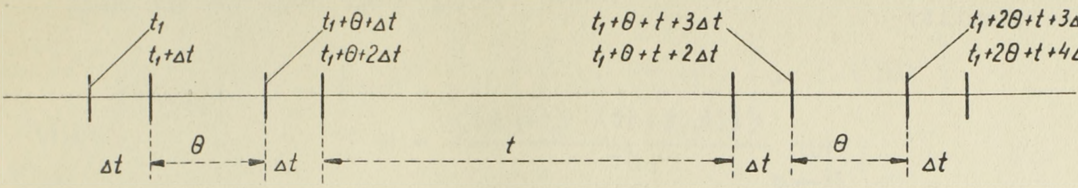
from which /4.1/ follows immediately. $\Theta \to \infty$ gives $D_\delta \to 2D(\Delta t)$ as could be expected.

It can be shown that the correlation between differences is weaker than that between the counts themselves. The method of Appendix I can easily yield the first-mentioned correlation function.

The succession of the counting intervals is shown in Fig.17. As

$$\mathcal{K}_\delta(t,\Theta,\Delta t) = \left\langle\left[\zeta(t_1,t_1+\Delta t) - \zeta(t_1+\Theta+\Delta t,t_1+\Theta+2\Delta t)\right]\times\right.$$

$$\left.\times\left[\zeta(t_1+\Theta+t+2\Delta t,t_1+\Theta+t+3\Delta t) - \zeta(t_1+2\Theta+t+3\Delta t,t_1+2\Theta+t+4\Delta t)\right]\right\rangle$$



<u>Fig.17</u> Spacing of the time intervals in the case of the differential method

we can write

$$\mathcal{K}_\delta(t,\Theta,\Delta t) = -\frac{1}{2} N\Delta t^2 \varepsilon \sum_{j=0}^{6} \omega_j D_j \left[1-\psi_j(\Delta t)\right]^2 \left[1-e^{-\omega_j(\Theta+\Delta t)}\right]^2 e^{-\omega_j t} \quad . \qquad /4.3/$$

For given Δt and t the smallest correlation will be found for $\Theta = 0$. In this case

$$\mathcal{K}_\delta(t,0,\Delta t) = -\frac{1}{2} N\Delta t^4 \varepsilon \sum_{j=0}^{6} \omega_j^3 D_j (1-\psi_j)^4 e^{-\omega_j t} \qquad /4.4/$$

which means a substantial reduction of the correlation for small values of Δt $(\omega_6 \Delta t < 1)$ in comparison with the correlation $\mathcal{K}(t,\Delta t)$. In the case if Δt is small and $\Theta = 0$ we obtain the approximate expression

$$D_\delta \sim 2N\Delta t \left\{1+\varepsilon D_6 \left[\psi_6 - \frac{1}{2}\alpha \Delta t (1-\psi_6)^2\right]\right\} \quad . \qquad /4.5/$$

The relaxation constant α and its error can be determined from the dependence on Δt of the measured data of $D_\delta / 2N\Delta t$ by means of the procedure discussed in connection with the direct method.

Appendix I

If $\zeta(t_1,t)$ is the number of counts in the interval (t_1,t) , the quantity

$$\lim_{\delta t \to 0} \frac{\langle \zeta(t_1,t+\delta t) - \zeta(t_1,t) \rangle}{\delta t} = N \qquad /I.1/$$

is called the counting rate. For the stationary case there is

$$\langle d\zeta(t) \rangle = \langle \zeta(t_1,t+dt) - \zeta(t_1,t) \rangle = N dt \ . \qquad /I.2/$$

Let us now consider the pairs of counts following one another after a lapse of time $t - t' = \Theta > 0$. If $t \neq t'$ the density of such pairs of events is given by

$$\lim_{dt,dt' \to 0} \frac{\langle d\zeta(t)\, d\zeta(t') \rangle}{dt\, dt'} = w(t-t') \ , \qquad /I.3/$$

while for $t = t'$ it becomes

$$\langle d\zeta(t) d\zeta(t') \rangle = \langle d\zeta(t) \rangle = N dt \ . \qquad /I.4/$$

The quantity we want to calculate is $\langle [\zeta(t_1,t)]^2 \rangle$. For this purpose let us divide the interval (t_1,t) into an integer number n of subintervals and denote the k-th of these subintervals by dt_k . It is obvious that

$$\zeta(t_1,t) = \sum_{k=1}^{n} [\zeta(t_1,t_k+dt_k) - \zeta(t_1,t_k)] = \sum_{k=1}^{n} d\zeta(t_k) \qquad /I.5/$$

$$(t = t_n + dt_n)$$

and hence we obtain

$$[\zeta(t_1,t)]^2 = \sum_{k=1}^{n} [d\zeta(t_k)]^2 + \sum_{k \neq k'} d\zeta(t_k) d\zeta(t_{k'}) \ . \qquad /I.6/$$

The quantity $\langle[\zeta(t_1,t)]^2\rangle$ can be calculated by averaging each of the terms on the right-hand side in /I.6/ and replacing the sum by an integral. Considering now that $\langle[d\zeta(t_k)]^2\rangle = \langle d\zeta(t_k)\rangle$ we can write

$$\langle[\zeta(t_1,t)]^2\rangle = \int_{t_1}^{t} N dt' + \int_{t_1}^{t} dt' \left[\int_{t_1}^{t'} w(t'-t'')dt'' + \int_{t'}^{t} w(t''-t')dt'' \right]$$

which can be reduced to a simpler form

$$\langle[\zeta(t_1,t)]^2\rangle = (t-t_1)N + \int_{t_1}^{t}\int_{t_1}^{t} w(|t'-t''|)dt'dt'' . \qquad /I.7/$$

It is possible to establish a relationship between the pair density function $w(t-t')$ and the correlation functions $\mathcal{K}(\Theta,\Delta t)$ and $\varkappa(\Theta,\Delta t)$ .Since $\mathcal{K}(\Theta,\Delta t)$ expresses the correlation between the counts in two time intervals having no common point, it is easily seen that

$$\mathcal{K}(\Theta,\Delta t) = \int_{t_1-\Delta t}^{t_1}\int_{t_2}^{t_2+\Delta t} w(t'-t'')dt'dt'' - N^2\Delta t^2 . \qquad /I.8/$$

$$(t_2-t_1 = \Theta > 0)$$

The function $\varkappa(\Theta,\Delta t)$ determines the correlation between the counts in two overlapping time intervals. Therefore taking into account that

$$\langle\zeta(t_1,t_2+\Delta t)\zeta(t_2,t_2+\Delta t)\rangle = N\Delta t + \int_{t_1}^{t_2+\Delta t}\int_{t_2}^{t_2+\Delta t} w(|t'-t''|)dt'dt'' , \qquad /I.9/$$

it follows that

$$\varkappa(\Theta,\Delta t) = N\Delta t - N^2\Delta t(\Delta t+\Theta) + \int_{t_1}^{t_2+\Delta t}\int_{t_2}^{t_2+\Delta t} w(|t'-t''|)dt'dt'' . \qquad /I.10/$$

Making use of the formula for $w(|t'-t''|)$ given by /3.6/ and /3.21/ it is possible to reproduce both the expressions $\mathcal{K}(\Theta,\Delta t)$ and $\varkappa(\Theta,\Delta t)$, derived previously by other methods.

Appendix II

The equation for $g(t,z)$ has already been derived in one of the earlier papers [4]. $g(t,z)$ satisfies the equation

$$\frac{\partial g(t,z)}{\partial t} = \alpha_c - (\alpha_c+\alpha_f)g(t,z)+\alpha_f\, q[\vec{x}(t,z)] \quad , \qquad /II.1/$$

where the components of the vector $\vec{x} = (x_0,\dots,x_6)$ are given by the relations

$$x_0 = \ln g(t,z)$$

$$x_i = \ln\left\{e^{-\lambda_i t} + \lambda_i \int_0^t e^{-\lambda_i t'} g(t-t',z)dt'\right\} . \qquad /II.2/$$

$$(i = 1,\dots,6)$$

The initial condition corresponding to /II.1/ has the following form: $g(0,z) = e^z$. On the basis of /2.13/ we get

$$\frac{dm(t)}{dt} = -\alpha m(t)+\alpha_f \sum_{i=1}^{6} \nu_i \lambda_i \int_0^t e^{-\lambda_i t'} m(t-t')dt' . \qquad /II.3/$$

The solution of this equation satisfying the initial condition $m(0) = 1$ can be written in the form:

$$m(t) = \sum_{j=0}^{6} c_j e^{-\omega_j t} \quad , \qquad /II.4/$$

where the relaxation constants $\omega_0,\dots,\omega_6$ are the roots of the equations

$$\frac{\beta^{eff}}{\ell} - \frac{\varrho}{\ell} = \omega + \frac{\beta^{eff}}{\ell\beta}\sum_{i=1}^{6}\frac{\lambda_i\beta_i}{\lambda_i-\omega} \quad ,$$

while the coefficients $c_0,\dots,c_6$ are given by /2.19/. The expression for $M(t)$ can immediately be calculated from the function $m(t)$ with help of the relation

$$M(t) = i_0 \int_0^t m(t')dt' . \qquad /II.5/$$

Appendix III

We shall prove that in the expansion of $C_1(t)$ in a power series in ε the first term is identical with $C(t)$, that is

$$C_1(t) = C(t) + o(\varepsilon) \; . \qquad /\mathrm{III}.1/$$

$C_1(t)dt$ is the probability that the time between two successive counts lies in the interval $(t, t+dt)$. In order to simplify the proof the effect of delayed neutrons will be neglected. The probability that the first count is recorded in the interval $(t'', t''+dt'')$ after a count recorded in the interval $(t', t'+dt')$ may be denoted by $w_1(t''-t')dt''dt'$. Since the probability of a count in the interval $(t', t'+dt')$ is known to be $\varepsilon \sum_{m'=0}^{\infty} m'P(m')dt'$ one can write

$$C_1(t''-t')dt'' = \frac{w_1(t''-t')\,dt''\,dt'}{\varepsilon \sum_{m'=0}^{\infty} P(m')m'dt'} \; , \qquad /\mathrm{III}.2/$$

where $P(m')$ represents the probability that m' neutrons are present in the reactor at an arbitrary time. Denote by $P(t''-t', 0, m|m'-1) = P(t,0,m|m'-1)$ the probability that there are m neutrons in the reactor at time t and no count was recorded in the time interval $(0,t)$, provided that $m'-1$ neutrons were present at time $t=0$. It can be readily seen that

$$C_1(t) = \frac{\varepsilon}{M_1} \sum_{m'=0}^{\infty} m'P(m') \sum_{m=0}^{\infty} m P(t,0,m|\, m'-1) \; . \qquad /\mathrm{III}.3/$$

Actually we have to determine the distribution function $P(t,0,m|\,m'-1)$. Since at time $t=0$ each of the $m'-1$ neutrons present in the reactor may start chain reactions independently of the others and in addition chain reactions may be initiated by source neutrons emitted in the interval $(0,t)$, the probability generating function

$$G(t,0,z|m'-1) = \sum_{m=0}^{\infty} e^{mz} P(t,0,m|m'-1) \qquad /III.4/$$

has the form

$$G(t,0,z|m'-1) = G(t,0,z)[g(t,0,z)]^{m'-1} , \qquad /III.5/$$

where

$$G(t,0,z) = \exp\left\{-i_0 \int_0^t [1-g(t',0,z)]dt'\right\} , \qquad /III.6/$$

while

$$g(t,0,z) = \sum_{m=0}^{\infty} e^{mz} p(t,0,m) . \qquad /III.7/$$

$p(t,0,m)$ is the probability that m neutrons are present in the reactor at time t and in the time interval $(0,t)$ no count was recorded, provided that at time $t=0$ one single neutron was present.

Using the procedure given in previous papers [4] for the probability generating function $g(t,0,z)$ the following equation can be derived:

$$\frac{\partial g(t,0,z)}{\partial t} = \alpha_c - (\alpha_c + \alpha_f + \varepsilon) g(t,0,z) + \alpha_f q[\ln g(t,0,z)] . \qquad /III.8/$$

It is to be noted that

$$g(0,0,z) = e^z \qquad /III.9/$$

and $g(t,0,0) = p(t,0) \leq 1$. $p(t,0)$ is the probability that no count is obtained in the interval $(0,t)$, provided that a neutron has been injected at time $t=0$ into the hitherto neutron-free system.

Let us introduce now the following notations:

$$g(t,0,0) = p(t,0) \quad , \qquad G(t,0,0) = P(t,0) \qquad /III.10/$$

and

$$\left[\frac{\partial g(t,o,z)}{\partial z}\right]_{z=0} = m(t,o) \qquad /III.11/$$

$$\left[\frac{\partial G(t,o,z)}{\partial z}\right]_{z=0} = P(t,o)M(t,o) .$$

It may be also noted that

$$M(t,o) = i_0 \int_0^t m(t',o)\,dt' . \qquad /III.12/$$

Taking into account the relations /III.10/ and /III.11/, we find

$$C_1(t) = \frac{\varepsilon}{M_1} P(t,o) \sum_{m'=0}^{\infty} m'P(m')\left[p(t,o)\right]^{m'-2}\left\{p(t,o)M(t,o)+(m'-1)m(t,o)\right\} . \qquad /III.13/$$

In order to determine the linear term of $C_1(t)$ in ε it is sufficient to know only that part of functions $p(t,o)$, $P(t,o)$, $m(t,o)$ and $M(t,o)$ which does not contain the parameter ε . We wish to find these functions in the form

$$\begin{aligned} p(t,o) &= \bar{p}_0(t) + \varepsilon \bar{p}_1(t) + o(\varepsilon) , \\ P(t,o) &= \bar{P}_0(t) + \varepsilon \bar{P}_1(t) + o(\varepsilon) , \\ m(t,o) &= \bar{m}_0(t) + \varepsilon \bar{m}_1(t) + o(\varepsilon) , \\ M(t,o) &= \bar{M}_0(t) + \varepsilon \bar{M}_1(t) + o(\varepsilon) . \end{aligned} \qquad /III.14/$$

Now we can derive from /III.8/ the following equations

$$\frac{dp(t,o)}{dt} = \alpha_c - (\alpha_c + \alpha_f + \varepsilon)p(t,o) + \alpha_f\, q\left[\ln p(t,o)\right] \qquad /III.15/$$

and

$$\frac{dm(t,o)}{dt} = -(\alpha_c + \alpha_f + \varepsilon)m(t,o) + \alpha_f \frac{m(t,o)}{p(t,o)} q'\left[\ln p(t,o)\right] . \qquad /III.16/$$

Taking into account the initial conditions $p(0,0) = m(0,0) = 1$ we find

$$\bar{p}_0(t) = 1 \quad , \qquad \bar{m}_0(t) = m(t) = e^{-\alpha t} \qquad /\text{III}.17/$$

and thus

$$\bar{P}_0(t) = 1 \quad , \qquad \bar{M}_0(t) = M(t) \ . \qquad /\text{III}.18/$$

Using the above results one can write

$$C_1(t) = \varepsilon \left[\frac{M_2 - M_1}{M_1} m(t) + M(t) \right] + o(\varepsilon) \ , \qquad /\text{III}.19/$$

where the first term on the right-hand side is actually $C(t)$. Thus we have shown that under certain conditions the assumption /3.29/ may be valid.

<u>Note</u> The evaluation of the second order term in $C_1(t)$ presents no particular difficulties, we find

$$C_1(t) = C(t) - \varepsilon^2 \Big\{ \bar{M}_1(t) + \bar{P}_1(t) M(t) + \frac{M_2 - M_1}{M_1} \big[\bar{m}_1(t) + \bar{p}_1(t) M(t) +$$

$$+ \bar{P}_1(t) m(t) \big] + \frac{M_3 - 3M_2 - 2M_1}{M_1} \bar{p}_1(t) m(t) \Big\} + o(\varepsilon^2) \ . \qquad /\text{III}.20/$$

It is of interest to note that the moment M_3 appears already in the second order term in ε .

A short computation will prove that

$$\bar{p}_1(t) = \frac{1 - e^{-\alpha t}}{\alpha} \quad , \qquad \bar{P}_1(t) = \frac{i_0 t}{\alpha} \left[1 - \frac{1 - e^{-\alpha t}}{\alpha t} \right] \ , \qquad /\text{III}.21/$$

$$\bar{m}_1(t) = \left[1 - \frac{\alpha_f}{\alpha} (\nu_{00} - 2\nu_0) \right] t e^{-\alpha t} + \frac{\alpha_f}{\alpha} (\nu_{00} - 2\nu_0) e^{-\alpha t} \frac{1 - e^{-\alpha t}}{\alpha} \ , \qquad /\text{III}.22/$$

and

$$\bar{M}_1(t) = \frac{i_0}{\alpha^2}\left[1-\frac{\alpha_f}{\alpha}(\nu_{00}-2\nu_0)\right]\left[1-e^{-\alpha t}(1+\alpha t)\right]+\frac{1}{2}\,\frac{i_0}{\alpha^2}\,\frac{\alpha_f}{\alpha}(\nu_{00}-2\nu_0)(1-e^{-\alpha t})^2. \qquad /\text{III}.23/$$

M_3 can be calculated from the generating function of $P(m)$.

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