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ON

MANY-SORTED LOGIC AS A TOOL FOR MODELLING

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DIFFERENT VALIDITY CONCEPTS IN MANY-SORTED LOGIC

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ABSTRACT

Many-sorted logic is used in several branches of computer science. This paper deals with a special feature of many-sorted logic: the problem of the so-called "empty-sorted" models. These models may have sorts with empty universes, too. The using of the classical validity relation in Tarski's sense for the class of empty-sorted models gives logical paradoxes. That is why we define a new validity concept which is based on an old idea of A. Mostowski. In this paper the detailed definition of both (Tarski's and Mostowski's) validity concepts are presented. The many-sorted language which is defined by the validity relation in Mostowski's sense, as it will be demonstrated, "works well". Los lemma and some axiomatizability theorems illustrating the advantages of this many-sorted language against the classical one are presented.

0. *INTRODUCTION*

Many-sorted logic is used in several branches of computer science. See e.g. Andreka-Nemeti [0]. Its mathematical formalism is applied for logical foundation of computer-aided problem solving, for definition of semantics of programming languages, in the theories of program verification and data bases, in knowledge representation, etc. The fundamental difference between many-sorted models and classical models is that the universes of many-sorted models are not homogeneous but consist of disjoint sets of different sorts. Thus, when defining the types of functions and relations, we must give not only the number of arguments but also the sort of every argument.

This paper deals with a special feature of many-sorted logic, namely with the problem of the so-called "empty-sorted" models. In most published works /e.g. Monk [9]/ all the models having a sort with empty universe are excluded. This exclusion restricts essentially the area where many-sorted logic can be used, that is why we omit this restriction. We introduce the class of *t-type normal models* /Mod_t/ which is identical with the class of many-sorted models defined in Monk [9], and we define the class of *t-type empty-sorted models* /Mod_t^o/, which contains Mod_t as a proper subclass: $Mod_t \subsetneq Mod_t^o$.

Using the classical validity relation in Tarski's sense /notation: $\models$ / for class of models Mod_t^o gives logical paradoxes. The reason of logical paradoxes is that the set of valuations of variables into a non-normal model is empty.

I. Nemeti suggested using an old idea of A. Mostowski [10] for many-sorted logic in order to avoid logical paradoxes. Mostowski utilized essentially the fact that the value of a formula in a model depends only on the free variables occurring in the formula. Tarski defined his valuation function in another way. The domain of a valuation function is the set of all variables /in general: ω/ independently of the number of free variables in the formula in question.

We introduce a new validity relation /notation: $\models$ / which is called validity relation in Mostowski's sense.

In this paper the detailed definitions of both validity concepts are presented. In order to be self-contained we give the definitions of many-sorted models and syntax of first-order many-sorted languages as well. A simple example is presented to show the difference between two validity concepts. Finally, we investigate some well-known theorems from the point of view of empty-sorted models.

1. NOTATION

Throughout the paper $\overset{d}{=}$ denotes the fact that the concept standing on the left-hand side of the symbol is defined by the expression standing on the right-hand side. For example, $\varphi \overset{d}{=} \psi$ means that x is equal to y by definition. Similarly, "$x \overset{d}{\Leftrightarrow} y$" means that formula φ is defined by formula ψ, and φ is defined to be true if and only if ψ is true. Throughout the paper "iff" is an abbreviation of "if and only if". Brackets (,) and [,] play the same role and they are used simultaneously.

The following notation is given for arbitrary sets.

$\cup A \overset{d}{=} \{x: (\exists y \in A) x \in y\}$.

$\cap A \overset{d}{=} \{x: (\forall y \in A) x \in y\}$.

$A \cup B \overset{d}{=} \cup\{A, B\}$.

$A \cap B \overset{d}{=} \cap\{A, B\}$.

$A \sim B \overset{d}{=} \{a \in A: a \notin B\}$.

<u>Natural numbers</u> are used in von Neumann's sense.

0 denotes the empty set.

$a + 1 \overset{d}{=} a \cup \{a\}$.

$\omega \overset{d}{=} \cap \{H: 0 \in H$ and $(\forall n \in H) n + 1 \in H\}$ and $(\forall n \in \omega) n \overset{d}{=} \{0, 1, \ldots, n-1\}$.

$|A|$ denotes the <u>cardinality</u> of the set A.

$Sb\ A \overset{d}{=} \{X: X \subseteq A\}$. $Sb\ A$ is the set consisting of all the subsets of A.

$Sb\ A$ is called the <u>power set</u> of the set A.

$(a, b) \overset{d}{=} \{\{a\}, \{a, b\}\}$ is the <u>ordered pair</u> of a and b, where the <u>first member</u> of the pair is a, and the second one is b.

Notation: $(a, b)_0 \overset{d}{=} a$ and $(a, b)_1 \overset{d}{=} b$.

$A \times B \overset{d}{=} \{(a, b): a \in A \text{ and } b \in B\}$. $A \times B$ is the <u>Cartesian product</u> of A and B.

$Dom\ A \overset{d}{=} \{a \in \cup\cup A: (\exists b)(a, b) \in A\}$. $Dom\ A$ denotes the <u>domain</u> of the set A.

$Rng\ A \overset{d}{=} \{b \in \cup\cup A: (\exists a)(a, b) \in A\}$. $Rng\ A$ denotes the <u>range</u> of the set A.

$A \upharpoonright B \overset{d}{=} A \cap (B \times Rng\ A) = \{(a, b) \in A: a \in B\}$. $A \upharpoonright B$ denotes the restriction of the set A to the set B.

Let f be an arbitrary set. f is a <u>function or a mapping or a sequence</u> iff all the elements of f are ordered pairs and $\forall a, b, c\ [((a, b) \in f \text{ and } (a, c) \in f) \to b = c]$.

If f is a function and $i \in Dom\ f$, then there exists exactly one set b such that $(i, b) \in f$.

b is said to be the value of the function f at the argument i and is denoted by

$f(i)$ or

$f\ i$ or

f_i.

${}^{A}B \overset{d}{=} \{f \in Sb(A \times B): f \text{ is a function}, Dom\ f = A\}$.

${}^{A}B$ denotes the <u>set of all the functions from A into B</u>.

$f: A \to B$ denotes that $f \in {}^{A}B$.

$f: A \rightarrowtail B$ denotes that $f \in {}^{A}B$ and f is a one-to-one mapping, i.e.

$$[f: A \rightarrowtail B] \overset{d}{\Leftrightarrow} [f: A \to B \text{ and } (\forall a, b \in A)(f(a) = f(b) \to a = b)].$$

$f: A \twoheadrightarrow B$ denotes that $f \in {}^{A}B$ and f is a <u>mapping from A onto B</u>, i.e. $Rng\ f = B$.

$f: A \rightarrowtail\!\!\twoheadrightarrow B$ denotes that $f \in {}^{A}B$ and f is a <u>one-to-one mapping from A onto B</u>.

Let $p_1, \ldots, p_n$ and S be fixed sets.

Let $\tau(x, p_1, \ldots, p_n)$ be an expression, which assigns a unique set denoted by $\tau(s, p_1, \ldots, p_n)$ to every $s \in S$. Then

$<\tau(s, p_1, \ldots, p_n)>_{s \in S} \stackrel{d}{=} <\tau(s, p_1, \ldots, p_n): s \in S> \stackrel{d}{=}$

$\{(s, \tau(s, p_1, \ldots, p_n)): s \in S\}$.

That is $<\tau(s, p_1, \ldots, p_n)>_{s \in S}$ is a function with the domain S.

If $n = 0$, i.e. there are no parameters $p_1, \ldots, p_n$ then

$<\tau(s)>_{s \in S} \stackrel{d}{=} <\tau_s>_{s \in S}$.

For example, suppose $\tau(s, p) \stackrel{d}{=} s \cap p$. Then $f \stackrel{d}{=} <s \cap p: s \in S>$ is a function for every fixed parameter p and S, otherwise f is not defined. That is the function f depends on the choice of the parameters p and S.

A further example: Suppose $S \stackrel{d}{=} \omega$ and $p \in \omega$. Then $g \stackrel{d}{=} <p + s: s \in \omega>$ is a function $g: \omega \to \omega$ and $(\forall x \in \omega)\ g(x) = p + x$. Obviously function g depends on the choice of the parameter p.

In particular, if f is a function and $Dom\ f = S$, then

$<f_s: s \in S> = f$,

$<f_s>_{s \in S} \stackrel{d}{=} <f_s: s \in S> = f$.

Let $n \in \omega$, and let f be a sequence of the length n.

The sequence f may be given by "enumeration" as follows:

$f = \langle f_0, f_1, \ldots, f_{n-1} \rangle$.

E.g. $f \overset{d}{=} \langle 5, 3, 8, 7 \rangle = \{(0, 5), (1, 3), (2, 8), (3, 7)\}$.

That is f is a sequence with length 4, $f\colon 4 \to \omega$ such that

$f(0) = f_0 = 5$, $f(1) = f_1 = 3$, $f_2 = 8$, $f_3 = 7$.

$S^+ \overset{d}{=} \bigcup\{{}^nS\colon n \in \omega \text{ and } n \neq 0\}$. S^+ denotes the set of all finite nonempty sequences of the elements of S.

Let A be a function. The <u>direct product</u> of A is as follows:

$$PA \overset{d}{=} \underset{i \in Dom\, A}{P} A_i \overset{d}{=} \{f \in {}^{Dom\, A}(\bigcup Rng\, A)\colon (\forall i \in Dom\, A) f_i \in A_i\}.$$

CONVENTION 0

Throughout the paper each symbol denotes a set unless it is declared to denote a class or a metaclass. All the notations introduced are used for classes and metaclasses as well as for sets in the usual way.

REMARK 0

Set theory, which is based on the hierarchy of sets - classes - metaclasses, is described e.g. in Herrlich-Strecher [6], where "conglomerate" is used instead of "metaclass". The main point of the hierarchy is

Sets $\subseteq$ *Metaclasses* such that *Sets* $\in$ *Metaclasses* and

$\langle Sets, \in \rangle \models ZFC$ and $\langle Metaclasses, \in \rangle \models ZFC$.

The difference between metaclasses and classes is that elements of a metaclass can be metaclasses classes or sets, while a proper class may have no elements but sets.

CONVENTION 1

1.1 From now on the ordered pairs and the 2-length sequences will not be distinguished. More exactly, $\langle x, y \rangle$ will denote both the ordered pair (x, y) and the function $\{(0, x), (1, y)\}$ for every set x and y, though they are not identical. The reason behind this convention is that it is not so important from the point of view of this paper, which meaning of the symbol $\langle x, y \rangle$ is to be considered. The only requirement is that condition

$$\forall x, y, u, w\ [\langle x, y \rangle = \langle u, w \rangle \Leftrightarrow (x = u \text{ and } y = w)]$$

holds for both meanings, and it obviously holds for both the ordered pairs and the 2-length sequences.

An important consequence of this convention is that $A \times A$ is identical with ${}^{2}A$ for every set A.

This convention (which is improper in principle) is very wide spread in mathematics, see e.g. Henkin-Monk-Tarski [5] p. 33, or Levy [7] Def. 4.15. p. 58. In these works one can also find the consequences of the convention above, and a technique which helps to avoid false results.

1.2 Let A be a set and $n \in \omega$. Then ${}^{n}A \times A$ is considered to be identical with ${}^{n+1}A$, i.e.

$${}^{n}A \times A = {}^{n+1}A \ .$$

Therefore the ordered pair $(<s_0, \ldots, s_{n-1}>, s_n)$ is considered to be identical with the sequence $<s_0, \ldots, s_{n-1}, s_n>$, and the Cartesian product is considered to be associative

$$(A \times B) \times C = A \times (B \times C) \subseteq {}^3(A \cup B \cup C).$$

Hence $A \times A \times A = {}^3A$.

DEFINITION 0 (n-ary relation, function)

Let B be a set and $n \in \omega$. By an <u>n-ary relation over B</u> we understand a set $R \subseteq {}^nB$, i.e. an n-ary relation is a set of sequences with the length n.

By an <u>n-ary function over B</u> we understand a set $f \in {}^{({}^nB)}B$. If f is an n-ary function, we write

$$f: {}^nB \to B.$$

□

COROLLARY 0

Due to Convention 1, <u>n-ary functions over B</u> are $n+1$-ary relations over B, since

$$\text{"}n\text{-ary functions over } B\text{"} = {}^{({}^nB)}B \subseteq ({}^nB) \times B = {}^{n+1}B.$$

This corollary is utilized essentially throughout the paper.

2. MANY-SORTED CLASSES OF MODELS

2.1. MANY-SORTED SIMILARITY TYPE

DEFINITION 1 (many-sorted similarity type)

A set t is said to be a many-sorted (or heterogeneous) similarity type if

$$t \in {}^3(Rng\ t) \quad \text{and}$$

$$t_1 : Dom\ t_1 \to (t_0)^+ \text{ and } t_2 \subseteq Dom\ t_1 .$$

□

NOTATION

Generally t_0 is denoted by S and t_2 by H, so

$$t = \langle S, t_1, H \rangle .$$

In Definition 1

$t_0 = S$	is the set of sorts,
t_1	is arity function,
$t_2 = H$	is the set of function symbols,
$Dom\ (t_1) \sim H$	is the set of relation symbols of the type t.

CONVENTION 2

From now on t denotes a many-sorted similarity type.

NOTATION

Let t be a similarity type and let $r \in Dom\ t_1$.

$$tr \overset{d}{=} t(r) \overset{d}{=} t_1(r) .$$

REMARK 1

If $r \in Dom\ t_1 \sim H$, i.e. r is a relation symbol, then $Dom\ (tr)$ is the number of the arguments of the relation symbol r. For example, let $t = <S,t_1,H>$ be a fixed similarity type such that

$S = \{p,q,k\}$, $t_1 = \{<r,<q,p,k>>, <f,<q,k>>\}$, $H = \{f\}$. $tr = <q,p,k>$.

Then $Dom\ (tr) = 3 = \{0,1,2\}$, and

$tr(0) = q$, $tr(1) = p$, $tr(2) = k$.

Let $n \overset{d}{=} Dom\ (tr)-1$.

If $f \in H$, i.e. f is a function symbol, then $n \overset{d}{=} Dom\ (tf)-1$ is the number of the arguments of the function symbol f .

2.2 MANY-SORTED MODELS

DEFINITION 2 (t-type model)

Let t be a many-sorted similarity type.

By a <u>many-sorted t-type model</u> we understand a pair $\mathfrak{A} = <A,R>$ iff the following (1)-(2) hold:

(1) A is a function such that

$Dom\ A = S$.

(2) R is a function, and conditions (i) - (ii) hold:

(i) $Dom\ R = Dom\ t_1$.

(ii) Let $r \in Dom\ t_1$ be an arbitrary symbol and

$n \overset{d}{=} Dom\ (tr)-1$. Then:

$R_r \subseteq \underset{i \leq n}{P} A_{tr(i)}$, i.e.

$R_r \subseteq A_{tr(0)} \times \dots \times A_{tr(n)}$.

Furthermore, if $r \in H$, then

$R_r: \underset{i<n}{P} A_{tr(i)} \rightarrow A_{tr(n)}$, i.e.

$R_r: (A_{tr(0)} \times \dots \times A_{tr(n-1)}) \rightarrow A_{tr(n)}$, i.e. relation R_r is a function with domain

$$Dom(R_r) = \underset{i<n}{P} A_{(t_1(r))_i} \quad .$$

□

By Definition 2 $\mathfrak{A}$ is a t-type many-sorted model iff

$$\mathfrak{A} = \langle\langle A_s\rangle_{s\in S}\ ,\ \langle R_r\rangle_{r\in Dom(t_1)}\rangle\ ,\ \text{i.e.}$$

$$\mathfrak{A}_0 = \langle A_s\rangle_{s\in S} \quad \text{and} \quad \mathfrak{A}_1 = \langle R_r\rangle_{r\in Dom(t_1)} \quad \text{and}$$

the conditions (1) and (2) above hold.

NOTATION

Let $\mathfrak{A}$ be an arbitrary t-type model and let $r \in Dom(t_1)$ be an arbitrary symbol. Then the set R_r is denoted alternatively by $r^{\mathfrak{A}}$, too. Thus

$$\mathfrak{A} = \langle A, R\rangle = \langle\langle A_s\rangle_{s\in S}\ ,\ \langle R_r\rangle_{r\in Dom(t_1)}\rangle \overset{d}{=}$$

$$\overset{d}{=} \langle\langle A_s\rangle_{s\in S},\ \langle r^{\mathfrak{A}}\rangle_{r\in Dom(t_1)}\rangle \ .$$

A_s $(s \in S)$ is said to be the universe of the sorts s , and

$$\mathfrak{A}_0 = A = \langle A_s \rangle_{s \in S}$$

is said to be <u>the system of universes of the model $\mathfrak{A}$</u>.

DEFINITION 3 (normal t-type model)

Let $\mathfrak{A}$ be a t-type model.

$\mathfrak{A}$ is a normal model iff $(\forall s \in S) A_s \neq 0$.

That is $\mathfrak{A}$ is a normal model if and only if there is no sort s such that the corresponding universe A_s is empty.

□

NOTATION

$Mod_t \overset{d}{=} \{\mathfrak{A} : \mathfrak{A}$ is a normal t-type model $\}$.

$Mod_t^0 \overset{d}{=} \{\mathfrak{A} : \mathfrak{A}$ is a t-type model $\}$.

Note that Mod_t and Mod_t^0 are not sets, but proper classes. $Mod_t \subsetneq Mod_t^0$, i.e. Mod_t is a proper subclass of class Mod_t^0 .

3. SYNTAX OF FIRST ORDER MANY-SORTED LANGUAGES

DEFINITION 4 (variables)

Let $t = \langle S, t_1, M \rangle$ be a similarity type and let $v : \omega \times S \rightarrowtail\!\!\!\!\twoheadrightarrow Rng\ v$ be a one-to-one function. Let set $Rng\ v$ be disjoint from any other set occurring in this paper, e.g. $Dom\ (t_1) \cap Rng\ v = 0$.

Let $\langle i, s \rangle \in \omega \times S$. Then

$$v_i^s \overset{d}{=} v(\langle i, s \rangle) .$$

v_i^s is called the <u>i-th variable of the sort s</u>.

Define

$$V^s \overset{d}{=} \{v_i^s : i \in \omega\} .$$

V^s is called the <u>set of variables of the sort s</u>.

Define

$$V \overset{d}{=} \bigcup_{s \in S} V^s .$$

V is said to be the <u>set of the variables</u>.

□

DEFINITION 5 (set of t-type terms : T_t)

Let $t = \langle S, t_1, H \rangle$ be a similarity type, and let V^s be a set of variables of the sort $s \in S$. Let G be the smallest sequence such that $Dom\ G = S$, and conditions (i) - (ii) hold:

(i) $(\forall s \in S)\ V^s \subseteq G(s)$.

(ii) Let $f \in H$ and $n = Dom(tf)-1$.

Suppose $(\forall i \in n)\ \tau_i \in G\ (tf(i))$. Then

$$f(\tau_0,\ldots,\tau_{n-1}) \in G(tf(n)) .$$

Obviously, there exists such a function G, and only one exists.

Let define

$$T_t^s \overset{d}{=} G\ (s)$$

for every $s \in S$.

T_t^s is said to be the <u>set of t-type terms of the sort s</u>.

Let $T_t \overset{d}{=} Rng\ G$, i.e. $T_t = \bigcup T_t^s$.

T_t is called the <u>set of t-type terms</u>.

□

DEFINITION 6 (set of t-type first order formulas : F_t)

The set of t-type <u>atomic formulas</u> is a set Af_t:

$Af_t \overset{d}{=} \{r(\tau_0,\ldots,\tau_n) : r \in Dom(t_1) \sim H,\ \ n = Dom(tr)-1$ and $\tau_i \in T_i^{tr(i)}$ for every $i \leq n\} \cup \{(\tau=\sigma) : \tau,\sigma \in T_t^s$ for $s \in S\}$.

The <u>set of t-type first order formulas</u> is the smallest set F_t such that

(i) $Af_t \subseteq F_t$.

(ii) Let $\varphi,\psi \in F_t$ and let $v_i^s \in V$ for any $s \in S$ and $i \in \omega$. Then

$$\{(\varphi\wedge\psi),\ \neg\varphi,\ \exists v_i^s\varphi\} \subseteq F_t .$$

□

CONVENTION 3

Let $\varphi, \psi \in F_t$ be arbitrary formulas and let $v_i^s \in V$ be a variable for any fixed $s \in S$ and $i \in \omega$. Then

$$(\varphi \lor \psi) \stackrel{d}{=} \neg(\neg\varphi \land \neg\psi),$$

$$(\varphi \rightarrow \psi) \stackrel{d}{=} (\neg\varphi \lor \psi),$$

$$(\forall v_i^s \varphi) \stackrel{d}{=} (\neg \exists v_i^s \neg\varphi).$$

4. *SATISFACTION AND VALIDITY RELATION IN TARSKI'S SENSE*

The concept *"satisfaction" in Tarski's sense* (notation: $\models$) is a 3-ary relation which connects a class of models, a set of formulas and the corresponding set of valuations. In the case of many-sorted logic, considering class of model Mod_t^o, set of formulas F_t and set of valuations $\underset{s\in S}{P}({}^{\omega}A_s)$ (see Def. 7 below), the satisfaction relation is:

$$Mod_t^o \times F_t \times \underset{s\in S}{P}({}^{\omega}A_s).$$

Let $\mathfrak{A} \in Mod_t^o$, $\phi \in F_t$, $k \in \underset{s\in S}{P}({}^{\omega}A_s)$.

Then $\models\langle\mathfrak{A}, \phi, k\rangle$ means, that the *valuation k satisfies the formula ϕ in the model $\mathfrak{A}$, or the formula ϕ is true in the model $\mathfrak{A}$ with respect to the valuation k.* Usually, we write $\mathfrak{A}\models\phi[k]$ instead of $\models\langle\mathfrak{A},\phi, k\rangle$, i.e.

$$\mathfrak{A}\models \phi\ [k] \overset{d}{=} \models \langle\mathfrak{A}, \phi, k\rangle$$

(see e.g. Andreka-Gergely-Nemeti [1] or Monk [9]).

By convention (sloppily), symbol $\models$ denotes the *validity relation in Tarski's sense*, too (see Monk [9]).

The validity is a binary relation, defined on a class of models and on a set of formulas. In our case:

$$\models\ \subseteq Mod_t^o \times F_t.$$

Thus the sequence of symbols $\mathfrak{A}\models\phi$ means, that the *formula ϕ is valid in the model $\mathfrak{A}$ or $\mathfrak{A}$ is a model of the formula ϕ.*

We define the satisfaction and the validity relation in Tarski's sense for many-sorted logic in details below.

DEFINITION 7 (valuation)

Let $\mathfrak{A} \in Mod_t^o$.

By a valuation of the variables into a model $\mathfrak{A}$ (shortly by a valuation) we understand a sequence of functions $k = \langle k_s \rangle_{s \in S}$ such that

$$(\forall_s \in S) k_s : \omega \to A_s \ .$$

That is $k_s \in {}^{\omega}(A_s)$ for every $s \in S$.

Therefore the set of all the valuations of the variables into $\mathfrak{A}$ is

$$\underset{s \in S}{P} \ ({}^{\omega}(A_s)).$$

□

DEFINITION 8 ($\tau^{\mathfrak{A}}[k]$).

Let $\tau \in T_t$, $\mathfrak{A} \in Mod_t^o$, $k \in \underset{s \in S}{P} \ ({}^{\omega}(A_s))$.

The meaning of the term τ in the model $\mathfrak{A}$ with respect to the valuation k (notation: $\tau^{\mathfrak{A}}[k]$) is defined by recursion

(i) If τ is a variable $v_i^s \in V^s$ ($s \in S$ and $i \in \omega$), then

$$v_i^{s\,\mathfrak{A}}[k] \overset{d}{=} k_s(i) \ . \qquad (k_s(i) \in A_s).$$

(ii) If τ is a term of the form $f(\tau_0, \ldots, \tau_{n-1})$, where $f \in H$, $n = Dom(tf)-1$ and $(\forall i \in n)[\tau_i \in T_t^{tf(i)}$ and $\tau_i^{\mathfrak{A}}[k]$ has already been defined] , then

$$f(\tau_0, \ldots, \tau_{n-1})^{\mathfrak{A}}[k] \overset{d}{=} f^{\mathfrak{A}}(\tau_0^{\mathfrak{A}}[k], \ldots, \tau_{n-1}^{\mathfrak{A}}[k]) \ .$$

□

DEFINITION 9 (satisfaction: $\mathfrak{A} \models \varphi\ [k]$*)*

Let $\varphi \in F_t$, $\mathfrak{A} \in Mod_t^o$, $k \in \underset{s \in S}{P} ({}^{\omega}A_s))$.

<u>"The valuation k satisfies the formula φ in the model $\mathfrak{A}$"</u> (notation: $\mathfrak{A} \models \varphi\ [k]$) is defined as follows:

1. <u>*Atomic formulas*</u>

(i) Let $\tau, \sigma \in T_t^s$. Then

$$\mathfrak{A} \models (\tau = \sigma)[k] \overset{d}{\Leftrightarrow} \tau^{\mathfrak{A}}[k] = \sigma^{\mathfrak{A}}[k].$$

(ii) Let $r \in Dom(t_1) \sim H$, $n = Dom(tr)-1$ and $(\forall i \subseteq n)\tau_i \in T_t^{tr(i)}$. Then

$$\mathfrak{A} \models r(\tau_o, \ldots, \tau_n)[k] \overset{d}{\Leftrightarrow} <\tau_o^{\mathfrak{A}}[k], \ldots, \tau_n^{\mathfrak{A}}[k]> \in r^{\mathfrak{A}}.$$

2. <u>*Formulas*</u>

Let $\varphi, \psi \in F_t$ and $v_i^s \in V^s$.

Suppose $\mathfrak{A} \models \varphi\ [k]$ and $\mathfrak{A} \models \psi\ [k]$ has already been defined. Then

(i) $\mathfrak{A} \models \neg\varphi\ [k] \overset{d}{\Leftrightarrow} (\mathfrak{A} \models \varphi\ [k]$ is not true).

(ii) $\mathfrak{A} \models (\varphi \wedge \psi)[k] \overset{d}{\Leftrightarrow} (\mathfrak{A} \models \varphi\ [k]$ and $\mathfrak{A} \models \psi\ [k])$.

(iii) $\mathfrak{A} \models \exists v_i^s\, \varphi[k] \overset{d}{\Leftrightarrow}$ (there exists a valuation $g \in \underset{s \in S}{P}({}^{\omega}(A_s))$ such that

$(\forall z \in S \sim \{s\})k_z = g_z$ and

$k_s \restriction (\omega \sim \{i\}) = g_s \restriction (\omega \sim \{i\})$ and

$\mathfrak{A} \models \varphi[g]$).

□

DEFINITION 10 (validity: $\mathfrak{A} \models \varphi$)

Let $\mathfrak{A} \in Mod_t^o$, $\varphi \in F_t$.

The formula φ is valid in the model $\mathfrak{A}$ or $\mathfrak{A}$ is a model of the formula φ iff

$$\mathfrak{A} \models \varphi \overset{d}{\Leftrightarrow} (\forall k \in \underset{s \in S}{P} ({}^{\omega}(A_s)))\, \mathfrak{A} \models \varphi\, [k].$$

□

5. *FIRST ORDER MANY-SORTED LANGUAGES WITH TARSKI'S VALIDITY RELATION*

DEFINITION 11. (L_t, L_t^o)

The triples

$$L_t \overset{d}{=} \langle F_t, Mod_t, \models \rangle \qquad \text{and}$$

$$L_t^o \overset{d}{=} \langle F_t, Mod_t^o, \models \rangle$$

are said to be first order many-sorted languages.

□

Note that both languages have the same syntax and validity relation, however, L_t is defined on the class of normal t-type models (Mod_t) and L_t^o is defined on the larger class of empty-sorted t-type models (Mod_t^o).

6. *SATISFACTION AND VALIDITY RELATION IN MOSTOWSKI'S SENSE*

Below we define a new validity relation which is different from that of Tarski. This validity relation is defined also by defining first satisfaction of formulas in models at valuations, but the definition of valuation is different from that of Tarski. Here the valuations depend on the formulas themselves. The crucial part of the definition of the satisfaction in Mostowski's sense is that one defines the set of valuations $val(\varphi,\mathfrak{M})$ for each formula φ and each model $\mathfrak{M}$. $val(\varphi,\mathfrak{M})$ gives evaluations only of those variables which freely occur in φ.

DEFINITION 12. $(var\ (\alpha))$.

Let $t = \langle S, t_1, H\rangle$ be a fixed similarity type. Let $\alpha \in T_t \cup F_t$, and $var(\alpha)_s \subseteq V^s$. $(s \in S)$. Let us denote the <u>set of free variables occurring in α</u> by $var(\alpha)$.

$$var(\alpha) \stackrel{d}{=} \langle var(\alpha)_s:\ s \in S\rangle \in {}^S(Sb\ \omega).$$

The definition of $var(\alpha)$ is given by a recursion below:

1. <u>Terms</u>

(i) Let $\alpha \in V^s$ be a variable of the sort s and let α be denoted by $v_i^s (i \in \omega)$. $var(v_i^s) \stackrel{d}{=} \langle 0:\ z \in S \sim \{s\}\rangle \cup \{\langle s, \{i\}\rangle\}$.

(ii) Let α be a term of the form $f(\tau_o, \ldots, \tau_n)$ where $f \in H$, $n = Dom(tf)-1$ and $(\forall i \in n)$ $(\tau_i \in T_t^{tf(i)}$ and $var(\tau_i)$ has already been defined). Then

$$var(f(\tau_o, \ldots, \tau_{n-1})) \stackrel{d}{=} \langle \cup\{var(\tau_i)_s:\ i \in n\}:\ s \in S\rangle.$$

2. Atomic formulas

(i) Let $\tau, \sigma \in T_t^s$.

$$var(\tau = \sigma) \stackrel{d}{=} var(\tau) \cup var(\sigma).$$

(ii) Let $r \in Dom(t_1) \sim H$ be a relation symbol, $n = Dom(tr)-1$ and $(\forall i \leq n)\ \tau_i \in T_t^{tr(i)}$.

Then

$$var(r(\tau_0, \ldots, \tau_n)) \stackrel{d}{=} \bigcup_{i \leq n} var(\tau_i).$$

3. Formulas

Let $\varphi, \psi \in F_t$ and $v_i^s \in V^s$.

Suppose $var(\varphi)$ and $var(\psi)$ have already been defined. Then

(i) $var(\neg\varphi) \stackrel{d}{=} var(\varphi)$.

(ii) $var(\varphi \wedge \psi) \stackrel{d}{=} var(\varphi) \cup var(\psi)$.

(iii) $var(\exists\, v_i^s\, \varphi) \stackrel{d}{=} var(\varphi) \sim \{\langle s, \{i\}\rangle\}$.

□

DEFINITION 13. $(val(\tau, \mathfrak{A}))$

Let $Tm_t \stackrel{d}{=} \bigcup \{T_t^s : s \in S\}$.

Let $\tau \in Tm_t$ and $\mathfrak{A} \in Mod_t^o$. Then

$$val(\tau, \mathfrak{A}) \stackrel{d}{=} P\langle {}^{var(\tau)(s)}A_s : s \in S\rangle.$$

□

DEFINITION 14. (set of valuations of formulas: $val(\varphi,\mathfrak{A})$*)*

Let $\varphi \in F_t$, $\mathfrak{A} \in Mod^o_t$.

Let us denote the <u>set of valuations of the formula φ into the model $\mathfrak{A}$</u> by $val(\varphi,\mathfrak{A})$.

$$val(\varphi,\mathfrak{A}) \overset{d}{=} P\langle {}^{var(\varphi)(s)}A_s:\ s \in S\rangle.$$

□

DEFINITION 15. $(\tau[k]^{\mathfrak{A}}_M)$

Let $\tau \in Tm_t$, $\mathfrak{A} \in Mod^o_t$, $k \in val(\tau,\mathfrak{A})$.

<u>The meaning of the term τ in the model $\mathfrak{A}$ with respect to the valuation k</u> (notation: $\tau[k]^{\mathfrak{A}}_M$) is as follows

(i) Let $\tau = v^s_i$, $v^s_i \in V^s$.

$$v^s_i[k]^{\mathfrak{A}}_M \overset{d}{=} k_s(i). \quad (k_s(i) \in A_s).$$

(ii) Let τ be a term of the form $f(\tau_o, \dots, \tau_{n-1})$ where $f \in H$, $n = Dom(tf)-1$ and suppose $[(\forall i \in n)\ (\tau_i \in T^{tf(i)}_t$ such that $(\forall g \in val(\tau_i,\mathfrak{A}))\ \tau_i[g]^{\mathfrak{A}}_M$ has already been defined$)]$.

Let $k \in val(f(\tau_o, \dots, \tau_{n-1}),\mathfrak{A})$.

Then

$$f(\tau_o, \dots, \tau_{n-1})[k]^{\mathfrak{A}}_M \overset{d}{=} f^{\mathfrak{A}}\langle \tau_i[\langle k_s \restriction var(\tau_i)_s:\ s \in S\rangle]^{\mathfrak{A}}_M:\ i \in n\rangle.$$

□

DEFINITION 16. (satisfaction in Mostowki's sense: $\mathfrak{A} \models \varphi[k]$)

$\varphi \in F_t$, $\mathfrak{A} \in Mod_t^o$, $k \in val(\varphi, \mathfrak{A})$.

The valuation k satisfies the formula φ in the model $\mathfrak{A}$ (notation: $\mathfrak{A} \models \varphi[k]$) is defined as follows:

1. Atomic formulas

(i) Let $\tau, \sigma \in T_t^s$.

$$\mathfrak{A} \models (\tau = \sigma)[k] \overset{d}{\Leftrightarrow} \tau[k_s \upharpoonright var(\tau)]_M^{\mathfrak{A}} = \sigma[k_s \upharpoonright var(\sigma)]_M^{\mathfrak{A}}.$$

(ii) Let $r \in Dom(t_1) \sim H$, $n \overset{d}{=} Dom(tr)-1$ and

$(\forall i \leq n)\tau_i \in T_t^{tr(i)}$. Then

$$\mathfrak{A} \models r(\tau_0, \ldots, \tau_n)[k] \overset{d}{\Leftrightarrow}$$

$$<\tau_0[k \upharpoonright var(\tau_0)]_M^{\mathfrak{A}}, \ldots, \tau_n[k \upharpoonright var(\tau_n)]_M^{\mathfrak{A}}> \in r^{\mathfrak{A}}.$$

2. Formulas

Let $\psi, \lambda \in F_t$ and $v_i^s \in V^s$.

Suppose $\mathfrak{A} \models \psi[g]$ and $\mathfrak{A} \models \lambda[h]$ have already been defined for all valuation $g \in val(\psi, \mathfrak{A})$ and $h \in val(\lambda, \mathfrak{A})$. Then

(i) $\mathfrak{A} \models \neg\psi[k] \Leftrightarrow$ ($\mathfrak{A} \models \psi[k]$ is not true).

(ii) $\mathfrak{A} \models (\psi \wedge \lambda)[k] \Leftrightarrow$ ($\mathfrak{A} \models \psi[k \upharpoonright var(\psi)]$ and $\mathfrak{A} \models \lambda[k \upharpoonright var(\lambda)]$).

(iii) $\mathfrak{A} \models \exists v_i^s \psi[k] \Leftrightarrow$ (there exists a valuation $g \in val(\psi, \mathfrak{A})$ such that $(k = g \upharpoonright (var(\psi) \sim \{<s, \{i\}>)$ and $\mathfrak{A} \models \psi[g]$)).

□

DEFINITION 17. (validity relation in Mostowski's sense: $\mathfrak{A} \models \varphi$*)*

Let $\varphi \in F_t$ and $\mathfrak{A} \in Mod_t^o$.

The formula φ *is valid in the model* $\mathfrak{A}$ *(*notation: $\mathfrak{A} \models \varphi$*)* is defined as follows:

$$\mathfrak{A} \models \varphi \overset{d}{\Leftrightarrow} (\forall k \in val(\varphi, \mathfrak{A}))\ \mathfrak{A} \models \varphi[k].$$

□

7. FIRST ORDER MANY-SORTED LANGUAGE WITH MOSTOWSKI'S VALIDITY RELATION

DEFINITION 18. (many sorted language L^{o}_{Mt})

The triple

$$L^{o}_{Mt} = \langle F_t, Mod^{o}_t, \models \rangle$$

is said to be <u>the first order many-sorted language with validity relation in Mostowski's sense</u>.

□

8. *EXAMPLE FOR DIFFERENT VALIDITY CONCEPTS*

Let $t = \langle S, t_1, H\rangle$ be a fixed similarity type such that $S = \{0, 1, 2\}$, $t_1 = \{\langle r, \langle 0, 2\rangle\rangle\}$, $H = 0$.

Let $\mathfrak{A} \in (Mod_t^0 \sim Mod_t)$ be a non-normal empty-sorted model, defined as follows (Fig. 1):

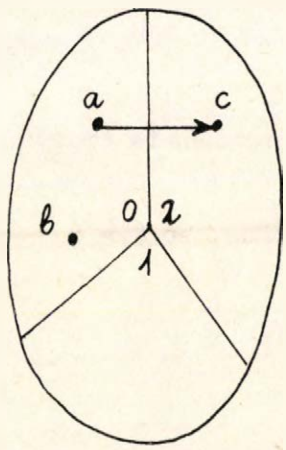

Figure 1

$\mathfrak{A} = \langle A, R\rangle = \langle\langle A_s\rangle_{s\in S}, \langle r^{\mathfrak{A}}\rangle_{r\in Dom(t_1)}\rangle$ such that

$A = \langle A_0, A_1, A_2\rangle$ where

$A_0 = \{a, b\}$, $A_1 = 0$, $A_2 = \{c\}$.

$R = \{\langle r, r^{\mathfrak{A}}\rangle\}$ where $r^{\mathfrak{A}} = \{\langle a, c\rangle\}$.

Consider the following formula φ: $r(v_1^0, v_1^2)$.

<u>Claim</u>

(i) $\mathfrak{A} \models \varphi$ i.e. the formula φ is valid in the model $\mathfrak{A}$ in Tarski's sense.

(ii) $\mathfrak{A} \not\models \varphi$ i.e. the formula φ is <u>not</u> valid in the model $\mathfrak{A}$ in Mostowski's sense.

PROOF of (i)

$\mathfrak{A} \models \varphi \overset{d}{\Leftrightarrow} (\forall k \in P(^{\omega}(A_s : s \in S)\mathfrak{A} \models \varphi[k].$

$P(^{\omega}(A_3): s \in S) = (^{\omega}A_0) \times (^{\omega}A_1) \times (^{\omega}A_2) =$

$= (^{\omega}\{a, b\}) \times (^{\omega}0) \times (^{\omega}\{c\}) =$

$= (^{\omega}\{a, b\}) \times 0 \times (^{\omega}\{c\}) = 0$, i.e.

the set of all valuation functions is empty. Thus

$(\forall k \in 0)\ \mathfrak{A} \models \varphi[k]$ is true, so $\mathfrak{A} \models \varphi$.

QED of (i).

REMARK 2

We can prove $\mathfrak{A} \models (\neg\varphi)$ in a similar way. It means that $\mathfrak{A} \models (\varphi \wedge \neg\varphi)$ which is a logical paradox.

PROOF of (ii)

$\mathfrak{A} \models \varphi \overset{d}{\Leftrightarrow} (\forall k \in P_{<}{}^{var(\varphi)(s)}A_s : s \in S>)\ \mathfrak{A} \models \varphi[k].$

$P_{<}{}^{var(\varphi)(s)}A_s : s \in S> = (^{\{1\}}\{a, b\}) \times (^{0}0) \times (^{\{1\}}\{c\}) =$

$= \{<1, a>, <1, b>\} \times \{0\} \times \{<1, c>\} \overset{d}{=} K.$

The set of all valuation function K has two elements

$K = \{k, g\}$ where $k = <<1, a>, 0, <1, c>>$

$g = <<1, b>, 0, <1, c>>$.

Remember, that in the present example φ is equivalent to the formula $r(v_1^0, v_1^2)$.

$r(v_1^0, v_1^2)[k] = r(a, c)$, and $r(a, c)$ is true, since $\langle a, c\rangle \in r^{\mathfrak{A}}$.

$r(v_1^0, v_1^2)[g] = r(b, c)$, and $r(b, c)$ is not true since $\langle b, c\rangle \notin r^{\mathfrak{A}}$.

So $\mathfrak{A} \not\models \varphi$.

QED of (ii).

9. *ŁOS LEMMA*

THEOREM 1 (generalization of Łos lemma)

1. Łos lemma holds in Mod_t with Tarski's validity, that is Łos lemma holds for $<F_t, Mod_t \models>$. In more details:

   Let I be an arbitrary set, $\mathfrak{A} \in {}^I Mod_t$, let U be an ultrafilter over I and let $\varphi \in F_t$ be an arbitrary formula. Then the following propositions (i) and (ii) hold:

   (i) $P\,\mathfrak{A}/U \models \varphi \Leftrightarrow (\exists Y \in U)(\forall i \in Y)\,\mathfrak{A}_i \models \varphi$.

   (ii) Let $k \in \underset{i \in I}{P}\ \underset{s \in S}{P}\ ({}^{\omega}A_{i,s})$, i.e. let $k_i \in \underset{s \in S}{P}\ ({}^{\omega}A_{i,s})$ be a valuation into $\mathfrak{A}_i$, i.e. $(\forall i \in I)(\forall s \in S)k_{i,s} \in {}^{\omega}A_{i,s}$.

   Let $(\forall s \in S)\overline{k_s}: \ \omega \to \underset{i \in I}{P}\ A_{i,s}/U$

   $$\overline{k_s} \overset{d}{=} \langle\langle k_{i,s}(n): \ i \in I\rangle/U \ : \ n \in \omega\rangle .$$

   Let $\overline{k} \overset{d}{=} \langle\overline{k_s}: \ s \in S\rangle$ be a valuation into $P\,\mathfrak{A}/U$. Then

   $$P\,\mathfrak{A}/U \models \varphi[\overline{k}] \Leftrightarrow (\exists Y \in U)(\forall i \in Y)\mathfrak{A}_i \models \varphi[k_i].$$

2. Łos lemma <u>does not hold</u> in general for $<F_t, Mod_t^o, \models>$, namely

   $|S| < \omega \Leftrightarrow$ Łos lemma holds for $<F_t, Mod_t^o, \models>$.

3. Łos lemma holds in Mod_t^o with Mostowski's validity, that is Łos lemma holds for $<F_t, Mod_t^o, \models>$.

PROOF

1. is proved as Theorem 3 in Markusz [8].

2. First we prove direction ⇐, that is we prove $|S| \geq \omega \Rightarrow$ Łos lemma does not hold for $\langle F_t, Mod_t^o, \models\rangle$.

Let $t = \langle\omega, 0, 0\rangle$.

Let $\mathfrak{A} \in {}^{\omega}Mod_t^o$ be, such that for every $n \in \omega$

$A_n = \langle A_{n,s}\rangle_{s\in\omega}$ where

$(\forall s \leq n)A_{n,s} = \{0\}$ and $(\forall s > n)A_{n,s} = 0$. See Fig. 2.

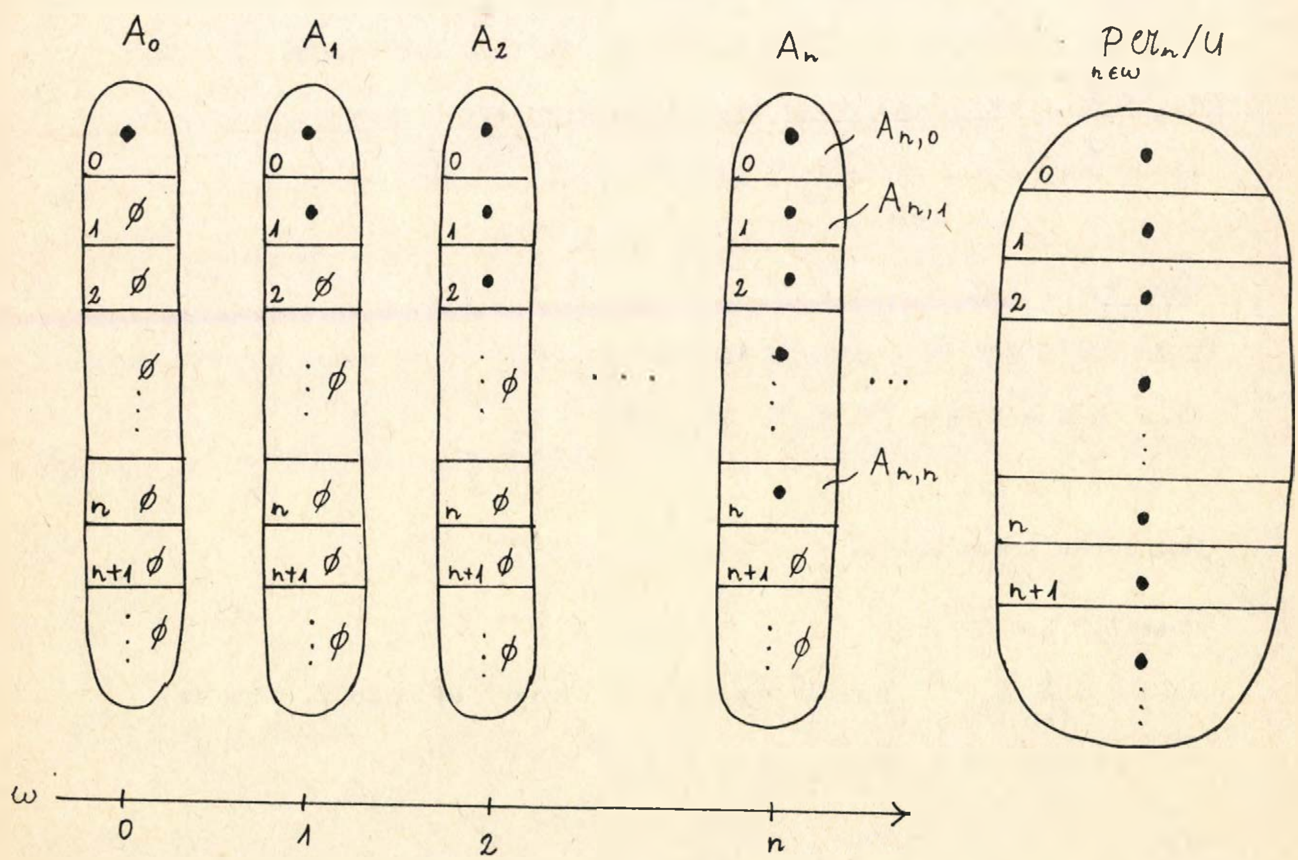

Figure 2

Let U be a nontrivial ultrafilter over ω. Then

$$P_{n\in\omega} \mathfrak{A}n/U = \langle\langle\{\overline{0}\}: s \in \omega\rangle, 0\rangle \quad \text{where}$$

$$\overline{0} = \langle 0, 0, 0, \ldots\rangle .$$

Now

$$P_{n\in\omega} \mathfrak{A}n/U \not\models \exists v_1^0 \; v_2^0 \; \neg(v_1^0 = v_2^0) \quad \text{and}$$

$$(\forall n \in \omega)\, \mathfrak{A}_n \models \exists v_1^0 \; v_2^0 \; \neg(v_1^0 = v_2^0) \quad \text{since any } \varphi \in F_t$$

is valid in Tarski's sense in every model which has at least an empty universe. We have proved direction $\Leftarrow$.

It remains to prove direction $\Rightarrow$. To this end assume $|S| < \omega$. Let $\mathfrak{B} \overset{d}{=} P_{i\in I} \mathfrak{A}_i/U$ be fixed with an ultrafilter U. Let $Y \overset{d}{=} \{i \in I: \bigwedge_{s\in S} A_{i,s} \neq 0\}$.

<u>Case 1</u>: $Y \notin U$.

Then $(\exists s \in S) B_s = 0$ since S is finite.

Then $Th\, \mathfrak{B} = F_t$ and $(\forall i \in I \sim Y)\ Th(\mathfrak{A}_i) = F_t$

$I \sim Y \in U$.

Hence Łos lemma holds.

<u>Case 2</u>: $Y \in U$

Then $\mathfrak{B} \cong P_{i\in Y} \mathfrak{A}_i/U^+$ where U^+ is the restriction of U to Y, that is $U^+ \overset{d}{=} \{X \in U: X \subseteq Y\}$.

Now no sort of $\mathcal{D}$ and $(\forall i \in Y)$ no sort of $\mathcal{M}_i$ is empty. Thus the classical version of Łos lemma can be applied e.g. using Andreka-Nemeti [3] or Markusz [8].

QED 2.)

3) Follows from Andreka-Nameti [3], e.g. see a similar proof in Andreka-Nemeti [4].

QED Theorem 1.

10. *THEOREMS OF AXIOMATIZABILITY*

10.1. *NOTATION*

Having two different validity concepts we should introduce new notation for the well-known metafunctions *Th* ("theory of") and *Mod* ("model of"). Let t be an arbitrary similarity type. Metafunctions Th^M and Mod^M are defined via Mostowsky validity $\models$;

$$(\forall K \subseteq Mod_t^0)\ Th^M(K) \stackrel{d}{=} \{\varphi \in F_t : K \models \varphi\}$$
$$(\forall T \subseteq F_t)\ Mod^M(T) \stackrel{d}{=} \{\mathfrak{A} \in Mod_t^0 : \mathfrak{A} \models T\}$$

and metafunction Mod^T and Th^T are de·ined via Tarki style validity $\models$:

$$(\forall K \subseteq Mod_t^0)\ Th^T(K) \stackrel{d}{=} \{\varphi \in F_t : K \models \varphi\}$$
$$(\forall T \subseteq F_t)\ Mod^T(K) \stackrel{d}{=} \{\mathfrak{A} \in Mod_t^0 : \mathfrak{A} \models T\} .$$

Note that Mod^T and Th^T are equivalent to metafunctions *Mod* and *Th*, respectively; see e.g. Markusz [8] .

Let $K \subseteq Mod_t^0$.
K is an EC_Δ^M iff $K = Mod^M Th^M K$ and
K is an EC_Δ^T iff $K = Mod^T Th^T K$.

Let $Y_t \subseteq F_t$ and $K \subseteq Mod_t^0$. We define metafunction $Y^M : Sb(Mod_t^0) \to Sb(Y_t)$ such that

$$(\forall K \subseteq Mod_t^0)\ Y^M(K) \stackrel{d}{=} Y_t \cap Th^M(K) .$$

Y^T is defined in a similar way and metafunctions Y^T and Y are equivalent.

We recall the definitions of sets of formulas Eq_t, Af_t, Qeq_t, Qaf_t, Ude_t, Uda_t, Uhf_t, $Unv_t \subseteq F_t$ (see Markusz [8]):

$Eq_t \overset{d}{=} \{\langle\tau,\sigma\rangle : \tau,\sigma \in T_t\}$ (equalities)

$Af_t \overset{d}{=} \{R(\tau_0,\dots,\tau_{t(R)-1}) : R \in Dom\ t_1 \sim t_2$ and $\tau_0,\dots,\tau_{t(R)-1} \in T_t\} \cup$
$\cup \{(\tau=\sigma) : \tau,\sigma \in T_t\}$. (atomic formulas)

$Qeq_t \overset{d}{=} \{(\bigwedge_{i<n} e_i \to e_n) : n \in \omega$ and $(\forall i \leq n) e_i \in Eq_t\}$.

(quasi-equalities)

$Qaf_t \overset{d}{=} \{\bigwedge_{i<n} R_i(\tau_{i,0},\dots,\tau_{i,t(R)-1}) \to R_n(\sigma_0,\dots,\sigma_k):$
$n,k \in \omega$, $(\forall i \leq n) R_i \in Dom\ t_1 \sim t_2$ and
$(\forall i \in n)(\forall j \in t(R_i)\tau_{i,j} \in T_t$ and $(\forall i \leq k)\ \sigma_i \in T_t\}$.

(quasi atomic formulas)

$Ude_t \overset{d}{=} \{\bigvee_{i<n} e_i : (\forall i < n)\ e_i \in Eq_t$ and $n \in \omega\}$.

(universal disjunction of equalities)

$Uda_t \overset{d}{=} \{\bigvee_{i<n} a_i : (\forall i < n) a_i \in Af_t$ and $n \in \omega\}$.

(universal disjunction of atomic formulas)

$Uhf_t \overset{d}{=} \{\bigvee_{i<n} \theta_i$: at most one of the formulas θ_i is an atomic formula
$(\forall i < n)\theta_i$ is an atomic formula or negation of atomic formula, $n \in \omega\}$.

(universal Horn formulas)

$Unv_t \overset{d}{=} \{\varphi \in F_t : \varphi$ is a formula without quantifier $\}$.

(universal formulas)

According to the definition of the metafunction Y^M, Eq^M, Af^M, Qeq^M, Qaf^M, Ude^M, Uda^M, Uhf^M, Uhv^M are also metafunctions.

The definitions of many-sorted operators

H_w	(weak homomorphic image)
H_s	(strong homomorphic image)
S_w	(weak submodel)
S_s	(strong submodel)
P	(direct product)
P^r	(reduced product)
Up	(ultraproduct)
Uf	(ultrafactor)

see in Markusz [8].

Let $K \subseteq Mod_t^0$.

$$P^+K = (PK \sim P\,O) \cup K .$$

We recall the definitions of the metafunctions

$$S_w^+ \text{ and } S_s^+ .$$

$$S_w^+ \overset{d}{=} \{Mod_t \cap S_w K : K \subseteq Mod_t\}$$

$$S_s^+ \overset{d}{=} \{Mod_t \cap S_s K : K \subseteq Mod_t\} .$$

10.2. *THEOREMS FOR NORMAL MODELS*

THEOREM 2 (axiomatizability theorems for normal models Mod_t with Tarski's style validity)

1) $Uf\ Up = Mod\ Th$

2) $H_w\ S_w^+\ P = Mod\ Eq$

3) $H_w\ S_s^+\ P = Mod\ Af$

4) $S_s^+\ P\ Up = Mod\ Qaf$

5) $S_w^+\ P\ Up = Mod\ Qeq$

6) $S_s^+\ P^+\ Up = Mod\ Uhf$

7) $H_w\ S_w^+\ Up = Mod\ Ude$

8) $H_w\ S_s^+\ Up = Mod\ Uda$

9) $S_s^+\ Up = Mod\ Unv$.

PROOF

The proof follows from Theorem 1 and 3 in Németi-Sain [11].

QED

10.3. *THEOREMS FOR EMPTY-SORTED MODELS*

THEOREM 3 (axiomatizability theorems for empty-sorted models Mod_t^0 with Mostowski style validity)

(i) $H_w\, S_s\, P^r = Mod^M\, Eq^M$

(ii) $S_s\, P^r = S_s\, P\, Up = Mod^M\, Qeq^M$

(iii) $H_w\, S_s\, Up = Mod^M\, Ude^M$

(iv) $S_s\, Up = Mod^M\, Unv^M$

PROOF

The proofs follows from Theorem 3 on p. 562 and Section 5 at the end of pp. 570 - 573 of Németi-Sain [11]. See also proofs in Sain [12].

QED

REMARK 3

By using Nemeti-Sain [11] axiomatizability theorems similar to Theorem 3 can be obtained for all the operators $H_i\, S_j\, P^r$, $S_j\, P^r$, $H_i\, S_j\, Up$, $S_j\, Up$ (with $i,j \in \{s,w\}$ arbitrary chosen) and the corresponding infinitary versions for $H_i\, S_j\, P$, $H_i\, S_i$ and also for $H_i\, S_j\, P^k$ where P^k denotes k-complete reduced products. Next we show that THEOREM 3 cannot be generalized to $\models$.

THEOREM 4

Let $i,j \in \{s,w\}$, and let $K \subseteq Mod_t^0$. $H_i\, S_j\, P^r\, K$, $S_i\, P^r\, K$, $H_i\, S_j\, Up\, K$ and $S_i\, Up\, K$ are not EC_Δ^T's (i.e. they are not axiomatizable in $\models$) for some K (this holds even for algebras).

PROOF

Let t be arbitrary such that $0,1 \in S$. Let $\mathfrak{A} \in Mod_t^0$ be such that $A_0 = 0$ and $A_1 = 2$. Then $Mod^T\, Th^T\{\mathfrak{A}\} = \{\mathfrak{B} : (\exists s \in S)\, B_s = 0\} \overset{d}{=} L$.
E.g. there is $\mathfrak{C} \in L$ with $C_0 = 3$ and $C_1 = 0$. Cleary $\mathfrak{C} \notin H_w\, S_w\, P^r\{\mathfrak{A}\}$.
(We note that $(\exists \mathfrak{N} \in P\{\mathfrak{A}\}\, (\forall s \in S) N_s = 1)$.

QED

PROPOSITION 5

Let $|S| > 1$. Then $S_s\, Up\, K$ is not EC_Δ^T for some K.

PROOF

Let $a,s \in S$ with $a \neq s$. Let $\mathfrak{A} \in Mod_t^0$ be such that $A_0 = A_s = 0$. Then
$\exists\, \mathfrak{B}, \mathfrak{C} \in Mod^T\, Th^T\{\mathfrak{A}\}$ such that $B_a = 2$, $C_s = 3$ and $B_s = C_a = 0$.
Clearly $\mathfrak{B}, \mathfrak{C} \notin S_s\, Up\, K$.

QED

THEOREM 5

Even if we assume $|S| < \omega$, the algebraic characterization of EC_Δ's as well as the Keisler-Shelah isomorphic ultrapower theorem fail for $\models$ (i.e. for Tarki style validity). In more detail:

Let t be arbitrary with $|S|>1$. Then

(i) $(\exists K \subseteq Mod_t^0)\ K = \mathcal{U}f\ \mathcal{U}p\ K \subsetneq Mod^T\ Th^T\ K$.

(ii) $(\exists \mathfrak{A}, \mathfrak{B} \in Mod_t^0)\ Th^T \mathfrak{A} = Th^T \mathfrak{B}$ but they have no isomorphic ultrapowers, i.e. $\mathcal{U}p\{\mathfrak{A}\} \cap \mathcal{U}p\{\mathfrak{B}\} = 0$.

Moreover, $\mathcal{U}f\ \mathcal{U}p\ \mathfrak{A} \cap \mathcal{U}f\ \mathcal{U}p\,\mathfrak{B} = 0$, too.

PROOF

(i) Let t and S as above.

Let $s,q \in S$ with $s \neq q$. (They exist by the assumption $|S| > 1$).

Let $K \stackrel{d}{=} \{\mathfrak{A} \in Mod_t^0 : A_s = 0$ and $A_q = 1\}$.

Then $\mathcal{U}f\mathcal{U}p\,K = K$. Since $Th^T K = F_t$ we have $(\exists \mathfrak{B}, \mathfrak{C} \in Mod^T ThK)\ B_q = 2$ and $C_s = 3$ and $B_s = 0$ and $C_q = 0$ are allowed. Clearly $\mathfrak{B} \notin K$.

QED of (i).

(ii) For these models we have $\mathfrak{B} \equiv_T \mathfrak{C}$ that is $Th^T \mathfrak{B} = F_t = T_h^T$. But

$\mathfrak{N} \in \mathcal{U}f\mathcal{U}p\ \mathfrak{B} \Rightarrow (N_s = 0$ and $|N_q| = 2)$ and

$\mathfrak{N} \in \mathcal{U}f\mathcal{U}p\ \mathfrak{C} \Rightarrow (|C_s| = 3$ and $C_q = 0)$.

Hence $\mathcal{U}f\mathcal{U}p\,\mathfrak{B} \cap \mathcal{U}f\mathcal{U}p\,\mathfrak{C} = 0$.

QED of (ii).

PROPOSITION 2

$H_s\ S_s\ P\ K$ and $H_w\ S_w\ P\ K$ are not axiomatizable (neither in $\models$ nor in $\models$) for some K. There is such a K without relation symbol, too. In other words, $H_s\ S_s\ P\ K$ is not an EC_Δ even for algebras.

PROOF

Completely analogous with that of Lemma 3 of Section 3 in Andreka-Németi [2]. Actually the quoted abstract model theoretic Lemma 3 implies the present proposition. Hint: Let t be arbitrary with S infinite and

$K \overset{d}{=} \{\mathfrak{A} \in Mod_t^0 : (\exists s \in S)\ A_s = 0\}$.

QED

11. *ACKNOWLEDGEMENTS*

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ON APPLICATION OF MANY-SORTED MODEL THEORETICAL OPERATORS IN KNOWLEDGE REPRESENTATION

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ABSTRACT

Finding the appropriate form of knowledge representation is an essential problem of most Computer-Aided Design (CAD), Computer-Aided Manufacturing (CAM) and expert systems. In this paper it is shown how the tools of many-sorted logic can be used for knowledge representation and a practical application of this method is presented. After giving precise mathematical definition of many-sorted models and many-sorted classes of models, we introduce some many-sorted operators such as weak and strong submodel, weak and strong homomorphic image and direct product. The main point of this paper is to show how one can give many-sorted operators practical (technical) meaning. All the abstract mathematical concepts are illustrated by practical examples from the area of production engineering in a house building factory. A small example shows, how naturally and easily one can transfer the knowledge represented by logical models to a PROLOG program using logic programming.

Keywords: knowledge representation, logic programming,
many-sorted logic, CAD/CAM.

0. INTRODUCTION

As the popularity of logic programming is increasing, more and more computer-aided/computer-manufacturing and expert systems are written in PROLOG or in other logic based programming languages [1]. In solving complex engineering problems the first and very important task is to find the form of representation of engineering knowledge. Among several knowledge representation tools (semantical networks, frames, etc.) it is mathematical logic which is the most appropriate to logic programming. In this paper we show how the many-sorted model theoretical concepts can be used for modelling certain engineering abstractions.

The fundamental difference between many-sorted models and classical logical models is that *the universes of many-sorted models are not homogeneous but consist of disjoint sets of different sorts*. Thus, when defining the types of functions and relations, we must give not only the number of arguments but also the sort of every argument. These models give us *better and finer modelling possibilities* than classical models [7].

Many-sorted logic is used not only for knowledge representation but in several other branches of computer science. Its mathematical formalism is applied e.g. for logical foundation of computer-aided problem solving [2], for definition of semantics of programming languages, in the theory of program verification [4] and of data bases [3]. Yet there are several areas of many-sorted model theory which are full of unsolved problems. One of them is the question of the so called "empty-sorted" models. In most published works (see [5]) all the models having a sort with empty universe are excluded. In this paper we omit this restriction, because these empty-sorted models can be well applied for knowledge representation (see Section 2.3). Another paper is to study the theoretical problems of the class of empty-sorted models [6].

This paper consists of two parts. In the first part (Section 1) *we define many-sorted models, normal and empty-sorted classes of models, and introduce some many-sorted operators* such as weak and strong submodel, weak and strong homomorphic image and direct product. In the second part (Section 2) *we show how one can give the abstract mathematical concepts practical meaning in knowledge representation of a CAD/CAM system*. In previous papers [7,8] we have introduced an architectural CAD program written in PROLOG. This program generates different versions of ground-plans of apartments according to the special needs of the customer. Then it designs a multistorey living-house. The architectural foundation of this program guarantees that these apartment houses can be built from prefabricated elements. The next step towards a CAD/CAM system is to design a production planning program for a house-building factory. All the examples taken from this area illustrate a new kind of knowledge representation tools and they could be very useful for a real application.

1. MANY-SORTED MODELS AND OPERATORS

DEFINITION 1 (*many-sorted similarity type*)

A set t is said to be a *many-sorted similarly type* if t is a triple $\langle t_0, t_1, t_2 \rangle$ where $t_1 : \mathrm{Dom}\ t_1 \to (t_0)^+$ and $t_2 \subseteq \mathrm{Dom}\ t_1$.

NOTATION

Generally t_0 is denoted by S and t_2 by H, so $t = \langle S, t_1, H \rangle$. In Definition 1

$t_0 = S$ denotes the set of the sorts $(t_0)^+ = (S)^+$ the set of all finite nonempty sequences of elements of S,

t_1	denotes the arity function,
$t_2=H$	denotes the set of function symbols,
$\mathrm{Dom}(t_1)\setminus H$	denotes the set of the relation symbols of type t.

REMARK

Let t be a similarity type and let $r\in\mathrm{Dom}t_1$

$$tr\overset{d}{=}t(r)\overset{d}{=}t_1(r) \ .$$

If $r\in\mathrm{Dom}t_1\setminus H$, i.e. r is a relation symbol, then $\mathrm{Dom}(tr)$ is the number of the arguments of the relation symbol r.

For example, let $t=\langle S,t_1,H\rangle$ be a fixed similarity type such that $S=\{p,q,k\}$, $t_1=\{\langle r,\langle q,p,k\rangle\rangle, \langle f,\langle q,k\rangle\rangle\}$, $H=\{f\}$. Then $tr=\langle q,p,k\rangle$, $\mathrm{Dom}(tr)=3\overset{d}{=}\{0,1,2\}$ and $tr(0)=q$, $tr(1)=p$, $tr(2)=k$.

Let $n\overset{d}{=}\mathrm{Dom}(tr)-1$. The natural number n denotes that the relation symbol r has $n+1$ argument, for $\mathrm{Dom}(tr)=n+1$. If $f\in H$, i.e. f is a function symbol, then $n\overset{d}{=}\mathrm{Dom}(tf)-1$ is the number of arguments of function symbol f.

DEFINITION 2 (many-sorted t-type model)

Let t be a many-sorted similarity type. By a *many-sorted t-type model* we understand a pair $\mathfrak{A}=\langle A,R\rangle$ such that (1), (2) hold:

(1) A is a function such that Dom A = S.

(2) R is a function,, and conditions (i), (ii) hold:

(i) Dom R = Dom t_1.

(ii) Let $r\in\mathrm{Dom}\ t_1$ be an arbitrary symbol and $n\overset{d}{=}\mathrm{Dom}(tr)-1$.
Then: $R_r\subseteq \mathop{P}_{i\leq n} A_{tr(i)}$, i.e. $R_r\subseteq A_{(tr)_0}\times\ldots\times A_{(tr)_n}$.
Furthermore, if $r\in H$, then
$R_r : \mathop{P}_{i<n} A_{tr(i)}\to A_{tr(n)}$, i.e. $R_r:(A_{(tr)_0}\times\ldots\times A_{(tr)_{n-1}})\to A_{(tr)_n}$
i.e. relation R_r is a function with domain
$\mathrm{Dom}\ (R_r) = \mathop{P}_{i<n} A_{(t_1(r))_i}$.

NOTATION

Let $\mathfrak{A}$ be an arbitrary t-type model, and let $r \in \mathrm{Dom}(t_1)$ be an arbitrary relation symbol. Then set R_r is denoted by $r^{\mathfrak{A}}$, too. Thus

$$\mathfrak{A} = \langle A, R \rangle = \langle \langle A_s \rangle_{s \in S}, \langle R_r \rangle_{r \in \mathrm{Dom}(t_1)} \rangle \overset{d}{=} \langle \langle A_s \rangle_{s \in S}, \langle r^{\mathfrak{A}} \rangle_{r \in \mathrm{Dom}(t_1)} \rangle .$$

A_s $(s \in S)$ is said to be the *universe of sort s* and $\mathfrak{A}_0 = A = \langle A_s \rangle_{s \in S}$ is said to be the *system of universes of model* $\mathfrak{A}$.

DEFINITION 3 (normal t-type model)

Let $\mathfrak{A}$ be a t-type model. $\mathfrak{A}$ is a *normal model* iff $(\forall s \in S) A_s \neq 0$. That is $\mathfrak{A}$ is a normal model if and only if there is no sort s such that the corresponding universe A_s is empty.

DEFINITION 4 (classes of many-sorted models)

$\mathrm{Mod}_t \overset{d}{=} \{ \mathfrak{A} : \mathfrak{A}$ is a normal t-type model$\}$.
$\mathrm{Mod}^o_t \overset{d}{=} \{ \mathfrak{A} : \mathfrak{A}$ is a t-type model$\}$.

Note that $\mathrm{Mod}_t \subsetneq \mathrm{Mod}^o_t$, i.e. class of *normal models* Mod_t is a proper subclass of class of *empty-sorted models* Mod^o_t

The syntax of the first order many-sorted language has similar rules as that of predicate calculus. The definition of set of first order many-sorted formulas F_t can be found in [5,9].

The connection between many-sorted formulas and models is defined by the "satisfaction" and "validity" relations. In this paper we use the validity relation in Tarski's sense ($\models \subseteq \mathrm{Mod}^o_t \times F^o_t$) defined in [9].

DEFINITION 5 (first order many-sorted language)

The triple $L_t = \langle F_t, Mod_t^o, \models \rangle$ is said to be a *first order many-sorted language.*

DEFINITION 6 (weak submodel)

Let $\mathfrak{A}, \mathfrak{B} \in Mod_t^o$ be two models. $\mathfrak{B}$ *is a weak submodel of model* $\mathfrak{A}$ (notation: $\mathfrak{B} \in S_w\{\mathfrak{A}\}$ or $\mathfrak{B} \subseteq_w \mathfrak{A}$) iff

(*i*) $(\forall s \in S) B_s \subseteq A_s$.
(*ii*) $(\forall r \in Dom(t_1)) r^{\mathfrak{B}} \subseteq r^{\mathfrak{A}}$.

DEFINITION 7 (strong submodel)

Let $\mathfrak{A}, \mathfrak{B} \in Mod_t^o$. $\mathfrak{B}$ *is a strong submodel of model* $\mathfrak{A}$ (notation: $\mathfrak{B} \in S_s\{\mathfrak{A}\}$ or $\mathfrak{B} \subseteq_s \mathfrak{A}$) iff

(*i*) $(\forall s \in S) B_s \subseteq A_s$.
(*ii*) $(\forall r \in Dom(t_1))[(n = Dom(tr) - 1) \rightarrow (r^{\mathfrak{B}} = r^{\mathfrak{A}} \cap \underset{i \leq n}{P} B_{tr(i)})]$.

DEFINITION 8 (homomorphism)

Let $\mathfrak{A}, \mathfrak{B} \in Mod_t^o$. By a *homomorphism* from $\mathfrak{A}$ into $\mathfrak{B}$ we understand a sequence of functions $f = \langle f_s \rangle_{s \in S}$ such that

(*i*) $(\forall s \in S)\ f_s : A_s \rightarrow B_s$.
(*ii*) $(\forall r \in Dom(t_1))\ (\forall \langle a_o, \ldots, a_n \rangle \in r^{\mathfrak{A}}) \langle f_{tr(o)}(a_o), \ldots, f_{tr(n)}(a_n) \rangle \in r^{\mathfrak{B}}$.

NOTATION

$f : \mathfrak{A} \rightarrow \mathfrak{B}$ denotes that f is a homomorphism from $\mathfrak{A}$ into $\mathfrak{B}$.

DEFINITION 9 (weak homomorphic image)

Let $\mathfrak{A}, \mathfrak{B} \in Mod^o_t$. $\mathfrak{B}$ is said to be a *weak homomorphic image* of model $\mathfrak{A}$
(notation: $\mathfrak{B} \in H_w\{\mathfrak{A}\}$) iff

(*i*) there exists a homomorphism $f:\mathfrak{A} \to \mathfrak{B}$.

(*ii*) $(\forall s \in S)\,\mathrm{Rng}\ f_s = B_s$.

DEFINITION 10 (strong homomorphic image)

Let $\mathfrak{A}, \mathfrak{B} \in Mod^o_t$ and $f:\mathfrak{A} \to \mathfrak{B}$ is said to be a *strong homomorphic image* of model $\mathfrak{A}$
(notation: $\mathfrak{B} \in H_s\{\mathfrak{A}\}$) iff

(*i*) $\mathfrak{B} \in H_w\{\mathfrak{A}\}$ (i.e. $\mathfrak{B}$ is a weak homomorphic image of $\mathfrak{A}$).

(*ii*) for every $r \in \mathrm{Dom}(t_1)$ if $n = \mathrm{Dom}(tr)-1$, then
$r^{\mathfrak{B}} = \{<f_{tr(o)}(a_o), \ldots, f_{tr(n)}(a_n)> : <a_o, \ldots, a_n> \in r^{\mathfrak{A}}\}$.

DEFINITION 11 (direct product)

Let I be an arbitrary set and $\mathfrak{A} \in {}^I Mod^o_t$. By the *direct product of models* $\mathfrak{A}_i$ ($i \in$ I) we understand a t-type model
$\mathfrak{B} = <<B_s>_{s \in S}, <r^{\mathfrak{B}}>_{r \in \mathrm{Dom}(t_1)}>$ such that

(*i*) $(\forall s \in S)\, B_s \overset{d}{=} P<A_{i,s} : i \in I>$.

(*ii*) $\forall r \in \mathrm{Dom}(t_1)$ if $n = \mathrm{Dom}(tr)-1$ then
$\forall b \in (B_{tr(o)} \times \ldots \times B_{tr(n)})[b \in r^{\mathfrak{B}} \Leftrightarrow (\forall i \in I) <b_o(i), \ldots, b_n(i)> \in r^{\mathfrak{A}_i}]$.

The direct product of models $\mathfrak{A}_i$ ($i \in$ I) is usually denoted by

$$P\mathfrak{A} \text{ or } \underset{i \in I}{P}\,\mathfrak{A}_i\ .$$

2. APPLICATION OF MANY-SORTED MODELS AND OPERATORS

2.1 Similarity type

Let us consider the world of a house building factory where prefabricated elements for apartment houses are to be produced. Our aim is to describe this world, formalize its rules and write a computer program which optimizes the production planning. We shall represent the world of this house building factory by a class of many-sorted models. The type t of the models is defined as follows: $t=<S,t_1,H>$ where

$S\overset{d}{=}\{\omega,fr,w,p,wb,d,sf,sw,sp,sd,a\}$

ω	- numbers,	sf	- sequence of front panels,
fr	- front panel,	sw	- sequence of wall panels,
w	- wall panel,	sp	- sequence of floor panels,
p	- floor panel,	sd	- sequence of doors and windows,
d	- door or window,	a	- apartment

are the sort of type t.

$H\overset{d}{=}\{f_1,f_2,f_3,f_4,f_5\}$ are the functions,
$\mathrm{Dom}\ t_1 \setminus H\overset{d}{=}\{r_1,r_2,r_3,r_4,r_5,r_6,r_7\}$ are the relations,

$$t_1\overset{d}{=}\{<f_1,<fr,sf,sf>>,<f_2,<w,sw,sw>>,<f_3,<p,sp,sp>>,$$
$$<f_4,<d,sd,sd>>,<f_5,<sf,sp,sd,sw,wb,a>>,$$
$$<r_1,<f_r,d>>,<r_2,<fr,d,d>>,<r_3,<fr>>,<r_4,<w,d>>,$$
$$<r_5,<w,d,d>>,<r_6,<w,w>>,<r_7,<fr,w>>\}.$$

Functions f_1, f_2, f_3, f_4 construct sequence of front panels, floor panels, windows and doors, respectively. Function f_5 constructs an apartment from the sequences of front panels, wall panels, floor panels, windows and doors, and a waterblock (Figure 1). Let us have a simple example with seven relations only, which, obviously, should be extended in the case of a real application.

r_1 - a front panel with one opening, connected with a window,
r_2 - a front panel with two openings, connected with two windows,
r_3 - a full front panel (without opening),
r_4 - a wall panel with one opening, connected with a door,
r_5 - a wall panel with two openings, connected with two doors,
r_6 - two wall panels have the same length,
r_7 - the length of a front panel (Figure 2).

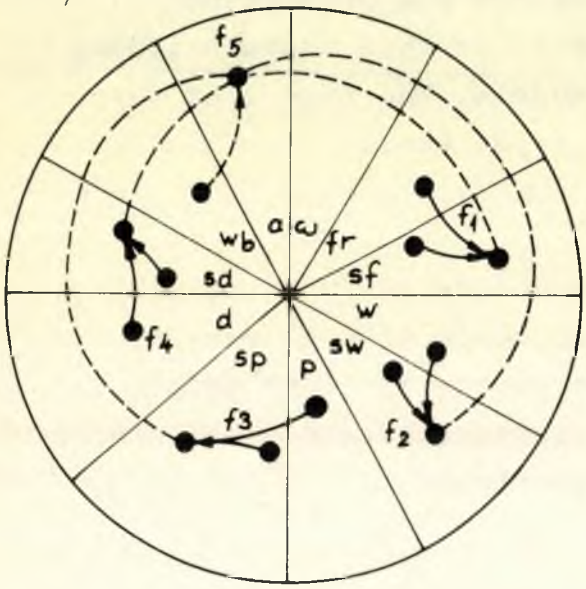

Figure 1. Many-sorted functions

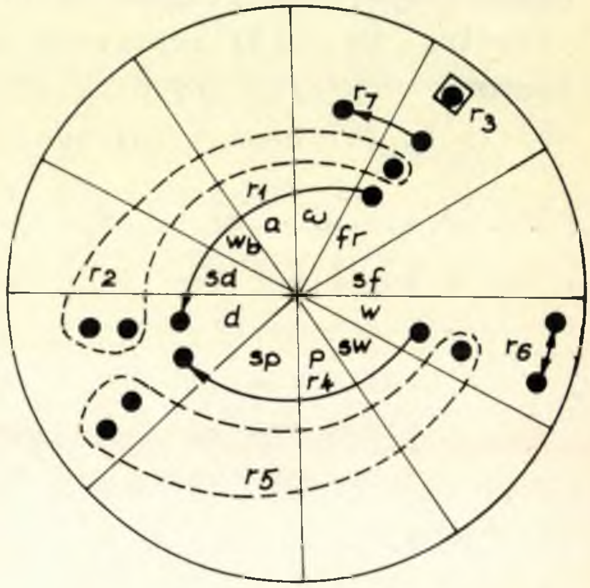

Figure 2. Many-sorted relations

2.2 Homomorphic image

Let us consider one of the apartment variants designed by our computer program in [8] (Figure 3). Figure 4 illustrates *all the prefabricated elements* (panels) which are needed for the apartment in Figure 3. The corresponding many-sorted model representing all the relations we know about these elements is denoted by $\mathfrak{A} \in \mathrm{Mod}_t^o$ in Figure 6. There are many elements in this apartment, which are quite alike. We should know which the *different elements* are.

The elements are different if their dimensions are different, or their dimensions are the same, but one of them has some openings for windows or doors, but the other has not. This "definition of being different" can be formalized by a homomorphism h from $\mathfrak{A}$ into a certain model $\mathfrak{B}$. This model $\mathfrak{B} \in \mathrm{Mod}^{o}_{t}$ has the elements we have defined as different elements, and all the relations holding in $\mathfrak{A}$ hold on the corresponding elements in $\mathfrak{B}$. It is easy to see by Definition 10 that $\mathfrak{B}$ is a *strong homomorphic image* of model $\mathfrak{A}$.

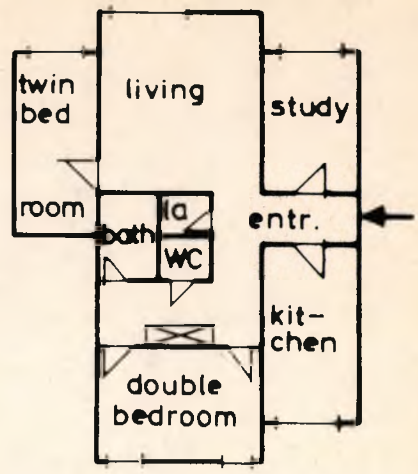

Figure 3. An apartment

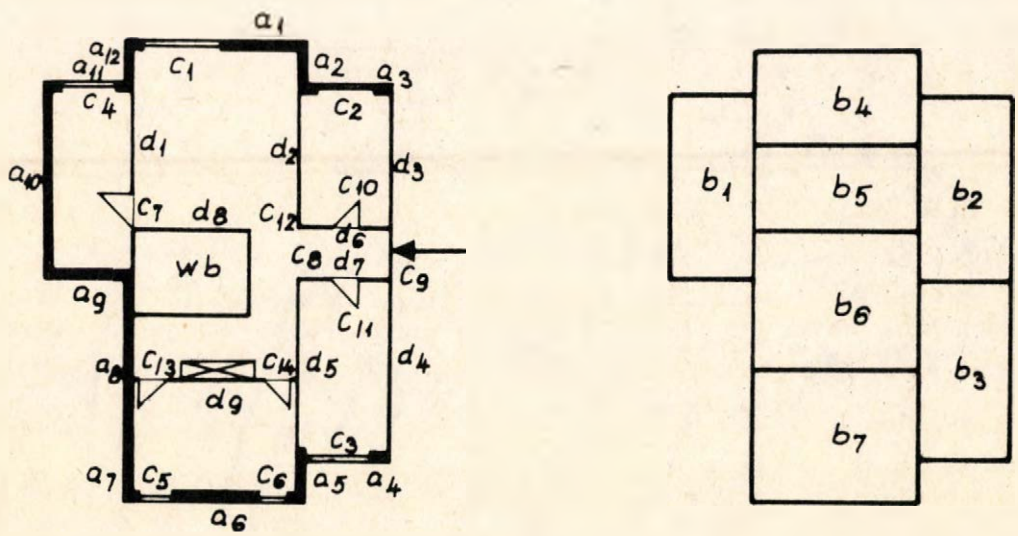

Figure 4. All the prefabricated elements for the apartment in Figure 3.

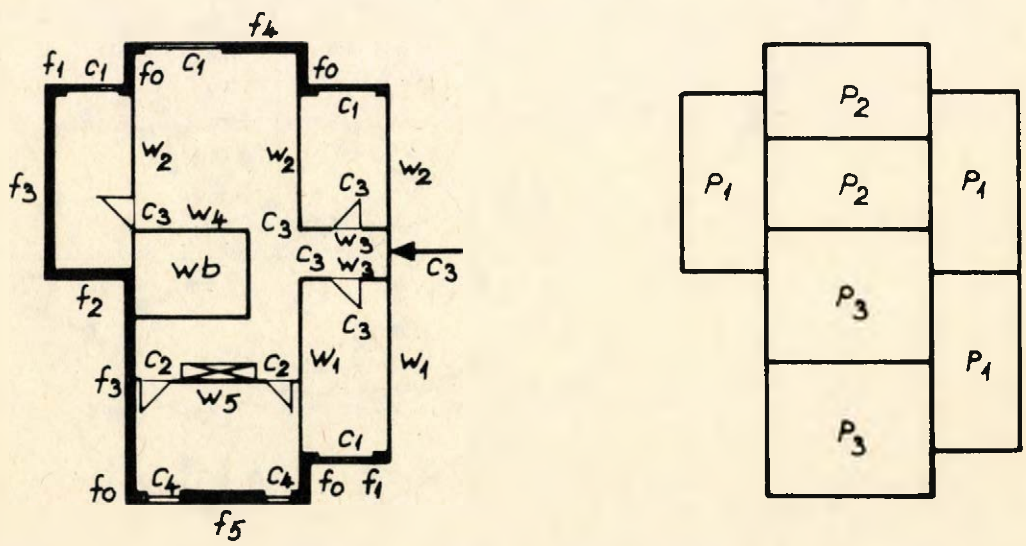

Figure 5. Different prefabricated elements for the apartment in Figure 3.

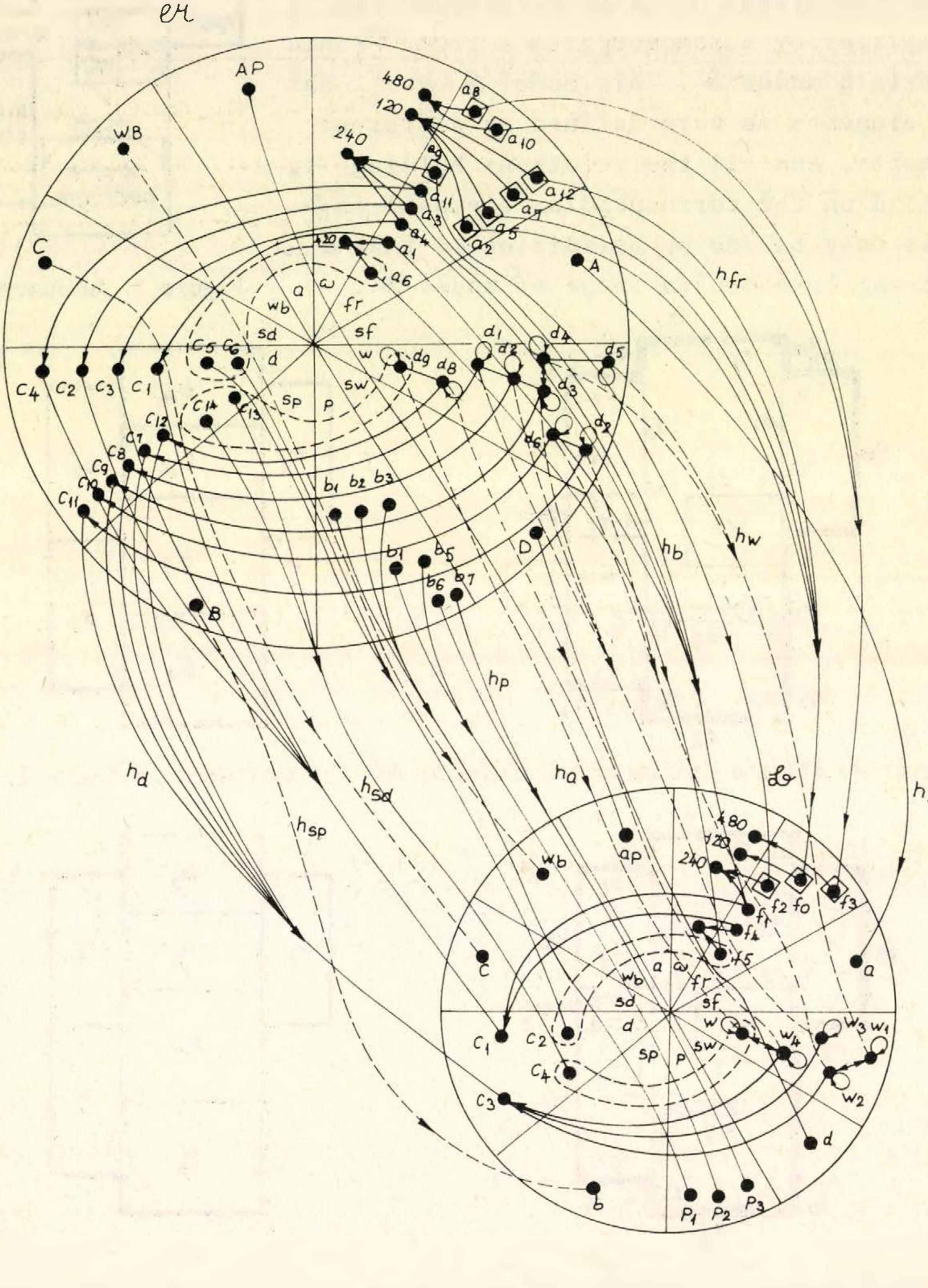

Figure 6. Homomorphic image

Let us formulate the concept of "being different" for the front panels in a many-sorted formula φ:

$$\rho\colon \forall\, v_1^{fr}\, v_2^{fr}\ ([\forall\, v_1^{\omega}\, v_2^{\omega}\ (\mathrm{length}(v_1^{fr}, v_1^{\omega}) \wedge \mathrm{length}(v_2^{fr}, v_2^{\omega}) \wedge \neg(v_1^{\omega} = v_2^{\omega}))] \vee$$
$$\forall\, v_1^{d}\ (\mathrm{full}(v_1^{fr}) \wedge [\mathrm{window}1(v_2^{fr}, v_1^{d}) \vee \mathrm{window}2(v_2^{fr}, v_1^{d}, v_1^{d})])] \rightarrow \neg(v_1^{fr} = v_2^{fr}))$$

where relations r_1, r_2, r_3, r_4 are denoted by window*1*, window*2*, full and length, respectively. A many-sorted variable is denoted by v_i^s ($s \in S$ and $i \in I$).

Obviously, $\mathfrak{B} \models \varphi$ and $\mathfrak{A} \not\models \varphi$. Generally, for every $\mathfrak{L} \in \mathrm{Mod}_t^o$, $\mathfrak{L} \models \varphi$, if there exists a model $\mathfrak{A} \in \mathrm{Mod}_t^o$ such that $\mathfrak{L} \in H_s\{\mathfrak{A}\}$.

We can represent formula φ in PROLOG programming language in a very natural way:

```
differ (X,Y) ← length (X,L1),
               length (Y,L2),
               L1 ≠ L2.
differ (X,Y) ← full (X), window (Y).
differ (X,Y) ← full (Y), window (X).

window (Z) ← window1 (Z,C1).
window (Z) ← window2 (Z,C2, C2).
```

Strings beginning with capital letters denote variables (e.g.L*1*), the other strings denote constants. Note that we used one-sorted variables, but this does not make any difference, since the pattern matching mechanism built into the deduction system of PROLOG automatically fulfils the requirement that only the terms of the corresponding sorts should be substituted for each other.

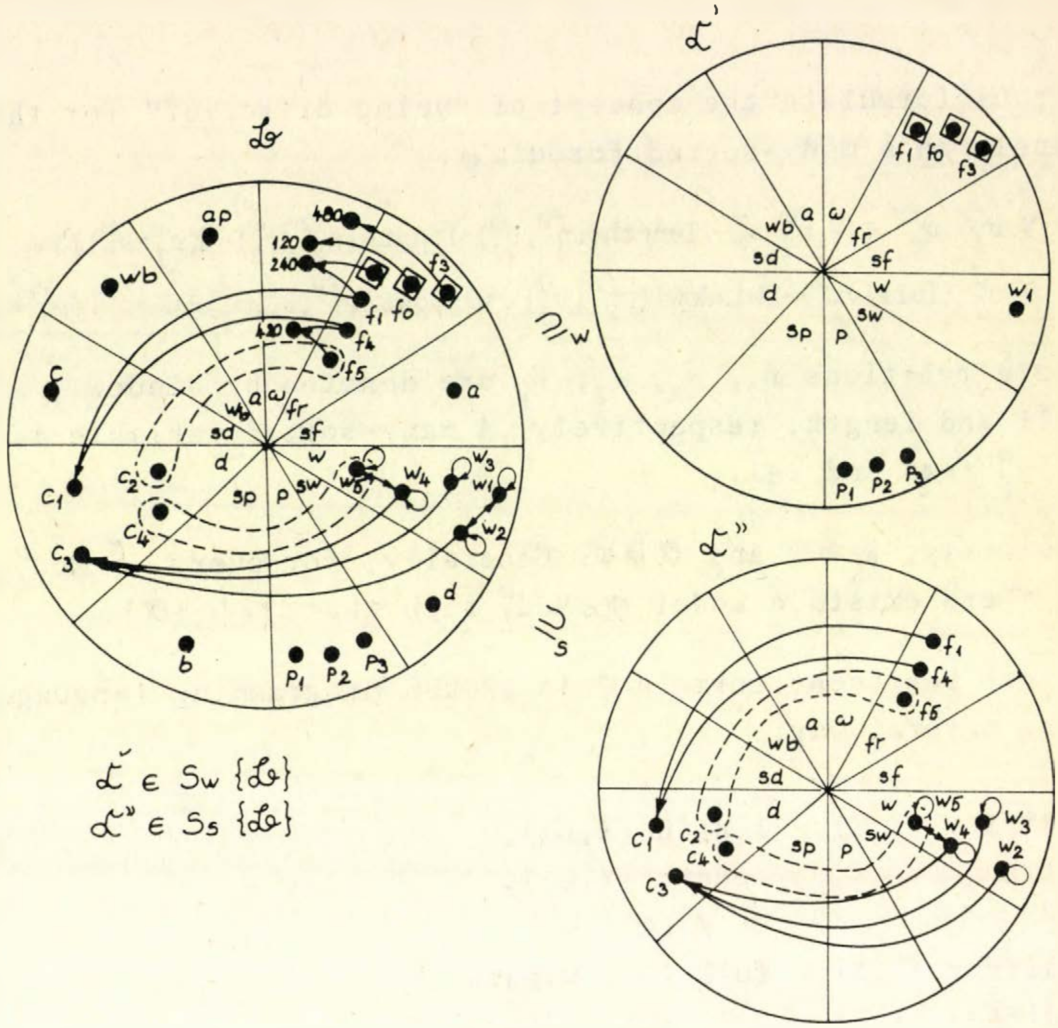

Figure 7. Weak and strong submodel

2.3 Submodel

Let us consider model $\mathcal{L}$ in Figure 6, i.e. the representation of all the different kinds of panels needed for the apartment in Figure 3. A production engineer in the house building factory should schedule the manufacturing of elements, considering the different technology of different production units. One unit can manufacture full panels only, and the other panel with openings for windows or doors. In Figure 7 model $\mathcal{L}' \in \mathrm{Mod}_t^o$ represents the elements produced by the first unit, and model $\mathcal{L}'' \in \mathrm{Mod}_t^o$ represents the elements produced by the second unit. Note that both models $\mathfrak{L}'$ and $\mathfrak{L}''$ are proper empty-sorted models, since e.g. C'_a C'_{wb}, C'_d are empty, and C''_a, C''_{wb} are also empty. It is easy to see, by Definitions 6, 7, that $\mathfrak{L}'$ is a weak submodel of $\mathfrak{B}$ and $\mathfrak{L}''$ is a strong submodel of $\mathfrak{B}$.

2.4 Direct product

Let us study the connection between wall panels and doors, considering three points of view: materials, manufacturing, and architecture. First of all we define a similarity type t' which is very simple: $t' \overset{d}{=} \langle S, t'_1, H\rangle$ where

$$S \overset{d}{=} \{w,d\},\ \ t'_1 \overset{d}{=} \{\langle r_6, \langle w,w\rangle\rangle, \langle r_4, w, d\rangle\rangle\},\ \ H \overset{d}{=} 0\ .$$

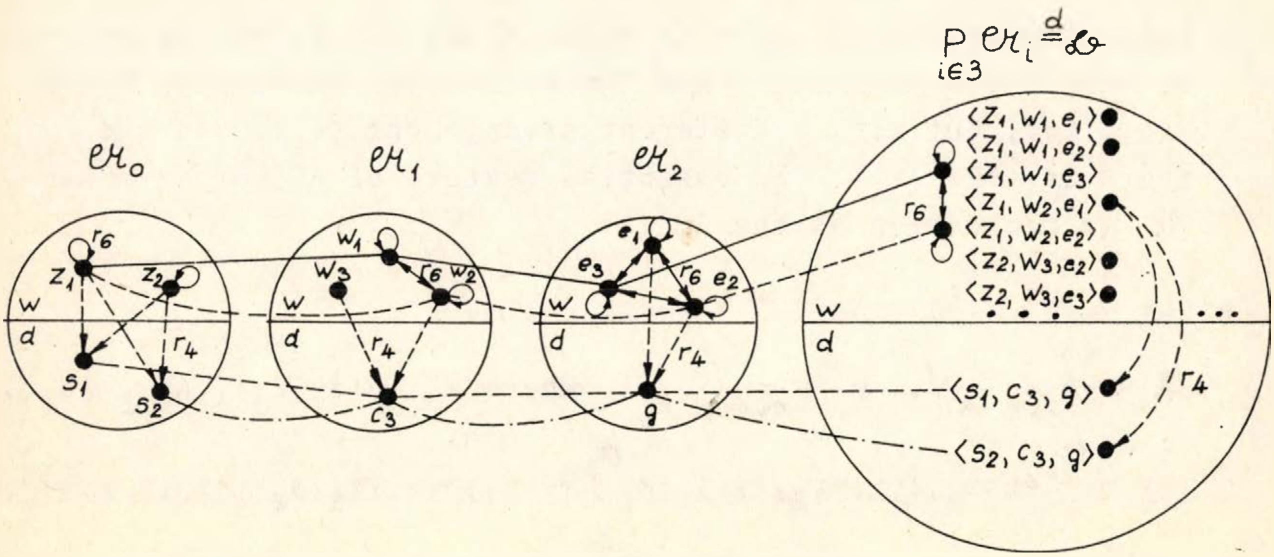

Figure 8. Direct product

1. *From the point of view of matherials* two wall panels are to be distinguished: concrete with strong reinforcement (z_1), and concrete with weak reinforcement (z_2). Doors are of two qualities: main door made of first class wood (s_1) and doors connecting rooms within the apartment, made of glass and second class wood (s_2). We define a model $\mathfrak{A}_0 \in \mathrm{Mod}^0_t$, which

represents these elements and the relations on them (Figure 8).

2. *From the point of view of production planning* we consider the dimensions of wall panels and their openings for doors. Thus in model $\mathfrak{A}_1 \in \mathrm{Mod}_t^o$, there are three elements of sort w: w_1, w_2, w_3. Wall panels w_1 and w_2 have the same dimensions (say big), and w_2 has an opening for a door, but w_1 has not. Element w_3 has different (say small) dimensions from that of w_1 and w_2. In model $\mathfrak{A}_1$ we have only one door. The essential feature of a door in this model is its dimensions.

3. *From the point of view of architectural design* we should consider the arrangement of openings for doors in wall panels. Each of them has the same dimensions but the first has an opening for a door (e_1), and the second has an opening for a door, too, but with a different arrangement (e_2), and the third is full (e_3). The essential feature of a door in model $\mathfrak{A}_2$ is the design of the door.

Let us define the model $\mathfrak{A}_o$ *precisely*:

$$\mathfrak{A}_o = \langle\langle A_{o,s}\rangle_{s\in S}, \langle r^{\mathfrak{A}_o}\rangle_{r\in Dom\, t_1'}\rangle \text{ where } A_{o,w}=\{z_1,z_2\},\ A_{o,d}=\{s_1,s_2\},$$

$$r_6^{\mathfrak{A}_o}=\{\langle z_1,z_1\rangle,\langle z_2,z_2\rangle\},\ r_4^{\mathfrak{A}_o}=\{\langle z_1,s_1\rangle,\langle z_2,s_2\rangle,\langle z_2,s_1\rangle,\langle z_2,s_2\rangle\}$$

The definitions of models $\mathfrak{A}_1$, $\mathfrak{A}_2$ can be given similarly (Figure 8). Let us see the direct product of models $\mathfrak{A}_o$, $\mathfrak{A}_1$, $\mathfrak{A}_2$:

$$\mathop{\mathrm{P}}_{i\in 3} \mathfrak{A}_i \overset{d}{=} \mathfrak{B} = \langle\langle B_s\rangle_{s\in S}, \langle r^{\mathfrak{B}}\rangle_{r\in Dom\, t_1'}\rangle \text{ where}$$

$B_w = \mathop{\mathrm{P}}_{i\in 3} A_{i,w}$, i.e. the set of all the sequences with length 3 of sort w (e.g. $\langle z_1, w_2, e_2\rangle$).

$B_d = \mathop{\mathrm{P}}_{i\in 3} A_{i,d}$, i.e. the set of all the sequences with length 3 of sort d (e.g. $\langle s_1, c_3, g\rangle$).

The elements of direct product $\mathcal{B}$ *represent complex information from all the three points of view at the same time.* For example: $q \stackrel{d}{=} \langle z_1, w_2, e_2 \rangle$ is a wall panel with strong reinforcement (z_1), with dimensions 4.8×2.7×0.25 (w_2), and with an opening for a big door, being in a distance of 1.2m from one side of the panel (e_2). $k \stackrel{d}{=} \langle s_1, c_3, g \rangle$ is a door, which is made of first class wood (s_1), with dimensions 0.95×1.96 (c_3), and with design denoted by g.

Direct product $\mathcal{B}$ *is a model representing all the possible t'-type relations on the complex objects of our world.* Let us have some example:

$$\langle\langle z_1, w_1, e_3\rangle, \langle z_1, w_2, e_2\rangle\rangle \in r_6^{\mathcal{B}}, \quad \text{since}$$

$$\langle z_1, z_1\rangle \in r_6^{\mathfrak{A}_0}, \quad \langle w_1, w_2\rangle\, r_6^{\mathfrak{A}_1}, \quad \langle e_3, e_2\rangle\, r_6^{\mathfrak{A}_2}.$$

But $\langle\langle z_2, w_2, e_3\rangle, \langle z_2, w_3, e_2\rangle\rangle \notin r_6^{\mathcal{B}}$, because in model $\mathfrak{A}_1$

$\langle w_2, w_3\rangle \notin r_6^{\mathfrak{A}_1}$ (see Definition 11).

Another example:

$$\langle\langle z_1, w_2, e_1\rangle, \langle s_1, c_3, g\rangle\rangle\, r_4^{\mathcal{B}} \quad \text{and}$$

$$\langle\langle z_1, w_2, e_1\rangle, \langle s_2, c_3, g\rangle\rangle\, r_4^{\mathcal{B}}.$$

It means that the wall panel $\langle z_1, w_2, e_1\rangle$ may have either a main door $\langle s_1, c_3, g\rangle$ or another door $\langle s_2, c_3, g\rangle$ (Figure 8).

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