Proceedings of the Colloquium on Abelian Groups

PROCEEDINGS
OF THE COLLOQUIUM
ON ABELIAN GROUPS

Edited by
L. Fuchs and *T. Schmidt*

In September 1963 at Tihany a one week Colloquium on the Theory of Abelian Groups was organized by the Hungarian Academy of Sciences and the János Bolyai Mathematical Society, cosponsored by the International Mathematical Union. From 12 countries, 48 invited mathematicians, among them the most eminent specialists in the theory of Abelian groups, took part in this Colloquium. Structural problems and homological methods of Abelian groups were the most important aspects discussed at the Tihany Colloquium. This volume comprises those studies which have not been published elsewhere.

AKADÉMIAI KIADÓ
PUBLISHING HOUSE OF THE
HUNGARIAN ACADEMY
OF SCIENCES
BUDAPEST

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TIHANY (HUNGARY), SEPTEMBER 1963

EDITED

by

L. FUCHS and E. T. SCHMIDT

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BUDAPEST, 1964

CONTENTS

PREFACE

This volume contains the papers presented at the Colloquium on Abelian Groups at Tihany, (Lake Balaton,) 2—7 September 1963. The papers which are published in full length elsewhere have been omitted, and a few papers which are closely connected with the subject of the Colloquium have been added.

The intention of the Organizing Committee of the Colloquium was to lay emphasis on homological methods and structural problems. Many talks were concerned with these topics. Since there was sufficient time, talks on different problems on Abelian groups and their applications could also be put in the program.

The Colloquium was organized by the Hungarian Academy of Sciences and the Bolyai Mathematical Society. It was sponsored by the Hungarian Academy of Sciences and the International Mathematical Union. Contributions to the costs of some of the participants were paid by their own universities and governments.

The Organizing Committee of the Colloquium consisted of Professors R. Baer, B. Charles, A. L. S. Corner, L. Fuchs, A. Kertész, L. Ya. Kulikov, A. G. Kuroš, R. S. Pierce, L. Rédei, E. Sąsiada, E. A. Walker, and Professors S. Mac Lane, B. Segre as representatives of the IMU. This committee drew up the list of invitations. The local organization was in the hands of a committee consisting of Professors E. Fried, L. Fuchs, G. Grätzer, E. T. Schmidt, O. Steinfeld, and staff members of the Academy (Mr. J. Kovács) and the Mathematical Society (Mrs. A. Pál, Miss A. M. Rónai).

The Organizing Committee owes a great debt of gratitude to all those who supported the Colloquium either financially or by rendering services, and who helped to arrange this important scientific meeting.

L. Fuchs

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THE SCIENTIFIC PROGRAMME

Morning, September 3

J. M. Maranda (Montreal, Canada): *Injective structures*

L. Fuchs (Budapest, Hungary): *Some generalizations of the exact sequences concerning Hom and Ext*

F. Loonstra (The Hague, Netherlands): *A-ordering of the group Ext (B, A)*

Afternoon, September 3

R. J. Nunke (Seattle, Wash., USA): *On the structure of the torsion product*

E. A. Walker (University Park, New. M., USA): *Applications of quotient categories to Abelian groups*

S. Balcerzyk (Toruń, Poland): *On classes of Abelian groups*

Morning, September 4

R. S. Pierce (Seattle, Wash., USA): *Some questions about primary groups*

R. A. Beaumont (Seattle, Wash., USA): *Quasi-isomorphism of p-groups*

Kin-ya Honda (Ikebukuro, Tokyo, Japan): *On the structure of Abelian p-groups*

Afternoon, September 4

L. Ya. Kulikov (Moscow, USSR): *On the structure of the group of extensions*

J. M. Irwin (University Park, New. M., USA): *On Σ^2-cyclic groups and Fuchs Problem* 5

A. Hulanicki (Wrocław, Poland): *On the factor group of the complete direct sum modulo the discrete direct sum*

Morning, September 5

W. Krull (Bonn, GFR): *Endomorphismenringe direkter Modulsummen*

E. Pogány (Budapest, Hungary): *On the endomorphism ring of Abelian p-groups*

A. L. S. Corner (Oxford, Great Britain): *Groups with prescribed endomorphism rings*

E. Sąsiada (Toruń, Poland): *Bemerkungen über das Problem von Whitehead*

Morning, September 6

E. S. Lyapin (Leningrad, USSR): *On the ordering of endomorphisms of certain p-groups*

A. D. Sands (Dundee, Great Britain): *Factorization of cyclic groups*

J. Rotman (Urbana, Ill., USA): *On the Grothendieck group of torsion-free groups of finite rank*

S. A. Khabbaz (New Haven, Conn., USA): *The role of tensor products in the splitting of Abelian groups*

Afternoon, September 6

M. I. Kargapolov (Novosibirsk, USSR): *Classification of ordered Abelian groups by elementary properties*

C. P. Walker (University Park, New. M., USA): *A generalization of purity*

F. Haimo (St. Louis, Mo., USA): *Word preserving subgroups of the symmetric group on the set of an Abelian group*

E. Fried (Budapest, Hungary): *Über Untergruppen Abelscher Gruppen, die in jedem Ringe Ideale sind*

Morning, September 7

B. Charles (Montpellier, France): *Méthodes topologiques en théorie des groupes abéliens*

J. de Groot (Amsterdam, Netherlands): *Additive groups of integer-valued functions over topological spaces*

A. Kertész (Debrecen, Hungary and Halle, GDR): *Ein Rangbegriff für Moduln*

V. Dlab (Prague, Czechoslovakia and Khartoum, Sudan): *A generalization of dependence relations*

The following papers were presented in the absence of the authors

P. J. Hilton (Ithaca, N. Y., USA): *Spectral sequences for factorizations of maps*

L. Procházka (Prague, Czechoslovakia): *Eine Bemerkung über die Spaltbarkeit Abelscher Gruppen*

L. Rédei (Szeged, Hungary): *Über die moderne Theorie der endlichen Abelschen Gruppen*

QUASI-ISOMORPHISM OF p-GROUPS[1]

By

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1. Introduction

The concept of quasi-isomorphism of Abelian groups has proved to be useful in the study of torsion-free groups. There is evidence that this concept may also be significant for torsion groups. It is therefore of interest to find necessary and sufficient conditions for two torsion groups to be quasi-isomorphic. In this paper, such conditions are found for countable torsion groups.

1.1. Definition. Let G and H be Abelian groups. Then G is quasi-isomorphic to H ($G \doteq H$) if there are subgroups $G' \subseteq G$, $H' \subseteq H$, and positive integers m and n such that $G' \cong H'$, $mG \subseteq G'$, and $nH \subseteq H'$.

It is routine to check that quasi-isomorphism is an equivalence relation and that if $G_i \doteq H_i$ for $i = 1, 2, \ldots, n$, then $\sum_{i=1}^{n} \oplus G_i \doteq \sum_{i=1}^{n} \oplus H_i$ [1; Lemmas 2.1 and 2.2]. Moreover, $G \doteq H$ if and only if there is a subgroup $G' \subseteq G$ and positive integers m and n such that $mG \subseteq G'$ and $G' \cong nH$.

The following proposition reduces the study of quasi-isomorphism of torsion groups to p-groups. Let G_p denote the p-primary component of the torsion group G.

1.2. Proposition. *Let G and H be torsion groups. Then $G \doteq H$ if and only if $G_p \cong H_p$ for almost all p and $G_p \doteq H_p$ for all p.*

Proof. If $G \doteq H$, then by 1.1, there are subgroups $G' \subseteq G$, $H' \subseteq H$ such that $G' \cong H'$ and G/G', H/H' are groups of bounded order. Since $G/G' \cong \sum_p \oplus G_p/G'_p$ and $H/H' = \sum_p \oplus H_p/H'_p$ are bounded, it follows that $G_p = G'_p$ and $H_p = H'_p$ for almost all p and that G_p/G'_p and H_p/H'_p are bounded for all p. Thus $G_p = G'_p \cong H'_p = H_p$ for almost all p and $G_p \doteq H_p$ for all p. Conversely, if $G = (\sum_{i=1}^{n} \oplus G_{p_i}) \oplus G_1$, $H = (\sum_{i=1}^{n} \oplus H_{p_i}) \oplus H_1$, where $G_1 \cong H_1$ and $G_{p_i} \doteq H_{p_i}$ for $i = 1, 2, \ldots, n$, it follows that $G \doteq H$.

[1] This work was supported by National Science Foundation (research grant No. GP 809).

By virtue of 1.2, the problem of deciding when two torsion groups are quasi-isomorphic is reduced to the question of when two primary groups are quasi-isomorphic. We will solve this problem for countable primary groups in terms of the Ulm invariants of the given groups.

Let $\lambda = \min \{\alpha \geq \omega \mid p^{\alpha+1} G = p^{\alpha} G\}$. For $\alpha < \lambda$, we will denote by $f_G(\alpha)$ the α-th Ulm invariant of the primary group G: $f_G(\alpha) = \dim (p^{\alpha} G \cap G[p]/p^{\alpha+1} G \cap G[p])$. We define $f_G(\lambda) = \dim (D[p])$, where D is the maximum divisible subgroup of G.

1.3. Theorem. *Let G and H be countable p-groups. Then $G \doteq H$ if and only if the following two conditions are satisfied:*

(I) *There exists an integer $k \geq 0$ such that for all integers $n \geq 0$ and $r \geq 0$*

$$\sum_{j=0}^{r} f_G(n+k+j) \leq \sum_{j=0}^{2k+r} f_H(n+j),$$

$$\sum_{j=0}^{r} f_H(n+k+j) \leq \sum_{j=0}^{2k+r} f_G(n+j);$$

(II) *$f_G(\alpha) = f_H(\alpha)$ for all $\alpha \geq \omega$.*

2. Proof of Theorem 1.3

The proof of the necessity of conditions (I) and (II) does not require that the groups G and H be countable. Moreover, several of the preliminary results used in the proof of the sufficiency of these conditions are completely general. Thus, in the statements of the following lemmas, countability is assumed only where it is required.

2.1. Proof of the necessity of (I) and (II). Suppose that the p-groups G and H are quasi-isomorphic. Then there exist groups $L_1 \subseteq G$, $L_2 \subseteq H$, and an integer $k \geq 0$ such that $p^k G \subseteq L_1$, $p^k H \subseteq L_2$, and $L_1 \cong L_2$. Since $L_1 \cong L_2$, $f_{L_1}(\xi) = f_{L_2}(\xi)$ for all ξ, and we write $f_L(\xi) = f_{L_1}(\xi) = f_{L_2}(\xi)$. For all $n \geq 0$,

(*) $$(p^{n+k} G)[p] \subseteq (p^n L_1)[p] \subseteq (p^n G)[p].$$

Hence for all $n \geq 0$ and all $r \geq 0$,

$$(p^n L_1)[p]/(p^{n+r+1} L_1)[p] \subseteq (p^n G)[p]/(p^{n+r+1} L_1)[p].$$

Moreover, there is an epimorphism of $(p^n G)[p]/(p^{n+k+r+1} G)[p]$ onto $(p^n G)[p]/(p^{n+r+1} L_1)[p]$. Consequently,

$$\sum_{j=0}^{r} f_L(n+j) = \dim((p^n L_1)[p]/(p^{n+r+1} L_1)[p]) \leq$$

$$\leq \dim((p^n G)[p]/(p^{n+k+r+1} G)[p]) = \sum_{j=0}^{k+r} f_G(n+j).$$

It also follows from (*) that for all $n \geq 0$ and all $r \geq 0$,

$$(p^{n+k} G)[p]/(p^{n+k+r+1} G)[p] \subseteq (p^n L_1)[p]/(p^{n+k+r+1} G)[p],$$

and there is an epimorphism of $(p^n L_1)\ [p]/(p^{n+k+r+1} L_1)\ [p]$ onto $(p^n L_1)[p]/(p^{n+k+r+1} G)[p]$. Therefore, as above

$$\sum_{j=0}^{r} f_G(n+k+j) \leq \sum_{j=0}^{k+r} f_L(n+j).$$

Similarly, using H and L_2, we obtain

$$\sum_{j=0}^{r} f_L(n+j) \leq \sum_{j=0}^{k+r} f_H(n+j), \quad \sum_{j=0}^{r} f_H(n+k+j) \leq \sum_{j=0}^{k+r} f_L(n+j).$$

Combining these two pairs of inequalities, we obtain the inequalities (I).

To prove (II), it is sufficient to observe that $p^\omega G = p^\omega L_1 \cong p^\omega L_2 = p^\omega H$. Therefore, $f_G(\alpha) = f_H(\alpha)$ for all $\alpha \geq \omega$.

We now wish to show that if G and H are countable p-groups which satisfy (I) and (II), then $G \doteq H$. Our first step is to derive a consequence of the inequalities (I).

2.2 LEMMA. *Suppose that $\{f_G(n)\}_{n<\omega}$ and $\{f_H(n)\}_{n<\omega}$ are sequences of non-negative integers such that $f_G(n) \neq 0$ for infinitely many n, $f_H(n) \neq 0$ for infinitely many n and both sequences satisfy* (I) *(with $k \geq 1$). For $m \geq 1$ define*

$$g_G^u(m) = \min \{l \geq u \mid m \leq \sum_{j=u}^{l} f_G(j)\},$$

$$g_H^u(m) = \min \{l \geq u \mid m \leq \sum_{j=u}^{l} f_H(j)\},$$

$$s_u = \sum_{j=u}^{u+k+1} f_H(j), \quad t_u = \sum_{j=u-k}^{u-1} f_H(j) \ \textit{for} \ u \geq k,$$

$$\triangle_n = \sum_{j=n-k}^{n+k} f_H(j) \quad \textit{for} \quad n \geq k.$$

Let $c \geq k$ be such that $\triangle_c \leq \triangle_n$ for all $n \geq k$. Then

$$g_H^c(m) - (4k+2) \leq g_G^c(m) \leq g_H^c(m) + (3k+2).$$

PROOF. We first prove that for all $n \geq k$ and all $m > t_u$,

$$\text{(III)} \qquad g_H^u(m - t_u) - k \leq g_G^u(m) \leq g_H^u(m + s_u) + k.$$

Let $g_H^u(m) = n$. Then by (I),

$$m \leq \sum_{j=u}^{n} f_G(j) \leq \sum_{j=u-k}^{n+k} f_H(j) = t_u + \sum_{j=u}^{n+k} f_H(j).$$

Therefore, $m - t_u \leq \sum_{j=u}^{n+k} f_H(j)$, and hence

$$g_H^u(m - t_u) \leq n + k = g_G^u(m) + k.$$

To prove $g_G^u(m) \leq g_H^u(m + s_u) + k$, we suppose first that $g_G^u(m) = n \leq \leq u + 2k$. In this case,

$$m + s_u > s_u = \sum_{j=u}^{u+k-1} f_H(j).$$

Hence, $g_H^u(m + s_u) \geq u + k$. Thus,

$$g_G^u(m) = n \leq u + 2k \leq g_H^u(m + s_u) + k.$$

Next, suppose that $g_G^u(m) = n > u + 2k$. Then by (I) we have

$$m > \sum_{j=u}^{n-1} f_G(j) \geq \sum_{j=n+k}^{n-1-k} f_H(j) = \sum_{j=u}^{n-1-k} f_H(j) - s_u.$$

Hence, $m + s_u > \sum_{j=u}^{n-1-k} f_H(j)$, so that $g_H^u(m + s_u) \geq n - k$. That is,

$$g_G^u(m) \leq g_H^u(m + s_u) + k,$$

which completes the proof of (III).

We next prove that for $1 \leq m \leq n$ and $n - m \leq \triangle_c$,

$$\text{(IV)} \qquad g_H^c(n) \leq g_H^c(m) + 2k + 2.$$

Let $g_H^c(m) = r$ and $g_H^c(n) = s$. Since $m \leq n$, it follows that $c \leq r \leq s$. We can suppose that $s - r \geq 2$. We have

$$\sum_{j=c}^{r} f_H(j) \geq m, \quad \sum_{j=c}^{s-1} f_H(j) < n.$$

Consequently

$$\sum_{j=r+1}^{s-1} f_H(j) < n - m \leq \triangle_c.$$

By the choice of c, if $\sum_{j=v}^{w} f_H(j) < \triangle_c$ for $k \leq v \leq w$, then $w - v \leq 2k$. Thus, $(s - 1) - (r + 1) \leq 2k$. That is, $g_H^c(n) - g_H^c(m) \leq 2k + 2$.

Now suppose that $t_c < m$. Since $m - (m - t_c) = t_c \leq \triangle_c$, it follows from (IV) that $g_H^c(m) \leq g_H^c(m - t_c) + 2k + 2$. Similarly, $g_H^c(m + s_c) \leq \leq g_H^c(m) + 2k + 2$. Hence by (III),

$$g_H^c(m) - (3k + 2) \leq g_G^c(m) \leq g_H^c(m) + (3k + 2).$$

For $1 \leq m \leq t_c$, we have

$$c \leq g_G^c(m) \leq g_H^c(m + s_c) + k \leq g_H^c(m) + 3k + 2.$$

(Note that for the second half of the inequality (III), $m > t_c$ is not required.) By the choice of c,

$$\sum_{j=c}^{c+2k} f_H(j) \geq \triangle_c \geq t_c > 0 .$$

Thus, $g_H^c(1) \leq c + 2k$. Moreover, since $m - 1 \leq t_c - 1 < \triangle_c$, it follows from (IV) that $g_H^c(m) \leq g_H^c(1) + 2k + 2$. Therefore, $g_H^c(m) \leq c + 4k + 2$. Hence,

$$g_H^c(m) - (4k + 2) \leq g_G^c(m) \leq g_H^c(m) + (3k + 2) .$$

2.3. Lemma. *Let G and H be p-groups which satisfy* (I) *and* (II). *Let l be a non-negative integer. Then $p\,G$ and p^lH satisfy* (I) *and* (II).

Proof. The proof follows from the fact that for any p-group $G, f_{p^lG}(n) = f_G(n + l)$.

2.4. Lemma. *Let G and H be countable p-groups which satisfy* (I) *and* (II) *and for which $f_G(n)$ and $f_H(n)$ are finite for almost all $n < \omega$. Then $G \doteq H$.*

Proof. We may assume that $f_G(n)$ and $f_H(n)$ are finite for all $n<\omega$. Indeed, for some $l \geq 0$, $f_{p^lG}(n)$ and $f_{p^lH}(n)$ are finite for all $n < \omega$, p^lG and p^lH satisfy (I) and (II) by Lemma 2.3, and $G \doteq H$ if and only if $p^lG \doteq p^lH$. We may further assume that $f_G(n) \neq 0$ for infinitely many $n < \omega$ and $f_H(n) \neq 0$ for infinitely many $n < \omega$. If $f_G(n) = 0$ for almost all n, then by (I), $f_H(n) = 0$ for almost all n. Hence $G = B \oplus D$, $H = B_1 \oplus D_1$, where B and B_1 are bounded and D and D_1 are divisible. It follows from (II) that $D \cong D_1$, so that $G \doteq H$. (The length of G = length of $H = \omega$ and $\dim(D[p]) = f_G(\omega) = f_H(\omega) = \dim(D_1[p])$.) We are now in the situation where the finite Ulm invariants of G and H satisfy the hypothesis of Lemma 2.2. (If G and H satisfy (I) with $k = 0$ and (II), then $f_G(\alpha) = f_H(\alpha)$ for all α and $G \cong H$.)

By Lemma 2.2, there exists $u \geq k$ and $M \geq 0$ so that

$$- M \leq g_G^u(n) - g_H^u(n) \leq M .$$

Consequently, there is an infinite sequence of integers $n_0 < n_1 < n_2 < \dots$ and an integer j such that

$$g_G^u(n_i) = g_H^u(n_i) + j$$

for all i. We may assume that $0 \leq j \leq M$. Let G_B and H_B be basic subgroups of G and H respectively. Then

$$G_B = \sum_{m<\omega} \oplus \{x_m\}, \quad H_B = \sum_{m<\omega} \oplus \{y_m\},$$

where $E(x_0) \leq E(x_1) \leq \dots, E(y_0) \leq E(y_1) \leq \dots$. Let

$$s = |\{i | E(x_i) \leq u\}| = \sum_{i<u} f_G(i) \quad \text{and} \quad t = |\{i | E(y_i) \leq u\}| = \sum_{i<u} f_H(i) .$$

Then,

$$E(x_{m+s}) = \min\{l | m + s \leq \sum_{j=0}^{l} f_G(j)\} + 1 = \min\{l | m + s \leq s + \sum_{j=u}^{l} f_G(j)\} + 1 =$$
$$= \min\{l \geq u | m \leq \sum_{j=u}^{l} f_G(j)\} + 1 = g_G^u(m) + 1\,.$$

Similarly, $E(y_{m+t}) = g_H^u(m) + 1$. Let $\omega - \{s + n_0, s + n_1, \ldots\} = \{l_0, l_1, \ldots\}$ where $l_0 < l_1 < \ldots$, and let

$$\omega - \{t + n_0,\ t + n_1, \ldots\} = \{m_0,\ m_1, \ldots\} \text{ where } m_0 < m_1 < \ldots\,.$$

Let

$$A = \sum_{i<\omega} \oplus \{x_{l_i}\},\ B = \sum_{i<\omega} \oplus \{y_{m_i}\}\,.$$

By Zippin's Theorem, there exist countable groups C and D such that $f_C(n) = f_G(n) - f_A(n)$, $f_D(n) = f_H(n) - f_B(n)$ for $n < \omega$, and $f_C(\alpha) = f_G(\alpha) = f_H(\alpha) = f_D(\alpha)$ for $\alpha \geq \omega$. By Ulm's Theorem,

$$G \cong A \oplus C\,,\quad H \cong B \oplus D\,.$$

Moreover, if C_B and D_B are basic subgroups of C and D respectively, then $C_B \cong \sum_{i<\omega} \oplus \{x_{s+n_i}\}$ and $D_B = \sum_{i<\omega} \oplus \{y_{t+n_i}\}$. Since

$$E(x_{s+n_i}) = g_G^u(n_i) + 1 = g_H^u(n_i) + 1 + j = E(y_{t+n_i}) + j\,,$$

it follows that $f_D(n) = f_C(n + j) = f_{p^jC}(n)$ for $n < \omega$, and $f_D(\alpha) = f_C(\alpha) = f_{p^jC}(\alpha)$ for $\alpha \geq \omega$. Hence, by Ulm's Theorem, $p^jC \cong D$, so that $C \stackrel{.}{\cong} D$. To complete the proof, it remains to be shown that $A \stackrel{.}{\cong} B$. Clearly, it is sufficient to show that $\sum_{i \geq s} \oplus \{x_{l_i}\} \stackrel{.}{\cong} \sum_{i \geq t} \oplus \{y_{m_i}\}$. Note that for $j \geq 0$, $l_{s+j} = s + r_j$ and $m_{t+j} = t + r_j$. Therefore,

$$|E(x_{l_{s+j}}) - E(y_{m_{t+j}})| = |g_G^u(r_j) - g_H^u(r_j)| \leq M\,.$$

Let $\sum_{j<\omega} \oplus \{z_j\}$ be a direct sum of cyclic groups such that

$$E(z_j) = \min\{E(x_{l_{s+j}}), E(y_{m_{t+j}})\}\,.$$

Then

$$E(z_j) \leq E(x_{l_{s+j}}) \leq E(z_j) + M,\ E(z_j) \leq E(y_{m_{t+j}}) \leq E(z_j) + M$$

for all j. Consequently, $\sum_{j<\omega} \oplus \{z_j\}$ is isomorphic to a subgroup $K \subseteq \sum_{j<\omega} \oplus \{x_{l_{s+j}}\}$ and to a subgroup $L \subseteq \sum_{j<\omega} \oplus \{y_{m_{t+j}}\}$ such that

$$p^M(\sum_{j<\omega} \oplus \{x_{l_{s+j}}\}) \subseteq K\,,\ p^M(\sum_{j<\omega} \oplus \{y_{m_{t+j}}\}) \subseteq L\,.$$

Thus, $\sum_{j<\omega} \oplus \{x_{l_{s+j}}\} \stackrel{.}{\cong} \sum_{j<\omega} \oplus \{y_{m_{t+j}}\}$, which completes the proof of the lemma.

2.5. Lemma. *Suppose that* $\{f_G(n)\}_{n<\omega}$ *and* $\{f_H(n)\}_{n<\omega}$ *are sequences of cardinal numbers which satisfy* (I). *If* $f_G(n)$ *is infinite for some* $n \geq k$, *then* *there exists* $m \geq 0$ *such that* $n-k \leq m \leq n+k$ *and* $f_G(n) \leq f_H(m)$.

PROOF. By (I), $f_G(n) \leq \sum_{j=n-k}^{n+k} f_H(j)$. Since $f_G(n)$ is infinite, it follows that

$$f_G(n) \leq \sum_{j=n-k}^{n+k} f_H(j) = \max\{f_H(j) | n-k \leq j \leq n+k\}.$$

2.6. LEMMA. *Let G and H be countable p-groups which satisfy* (I) *and* (II) *and suppose that* $f_G(n) = \aleph_0$ *for infinitely many* n. *Then*

$$G = G_1 \oplus G_2, \quad H = H_1 \oplus H_2,$$

where

(i) $G_1 \doteq G_2$,

(ii) G_2 *and* H_2 *are direct sums of cyclic groups,*

(iii) $f_{G_2}(n) = f_G(n)$ *and* $f_{H_2}(n) = f_H(n)$ *for* $n < \omega$.

PROOF. Let $N_l = \{n \geq k \mid f_G(n) = \aleph_0, f_H(n-l) = \aleph_0\}$. By Lemma 2.5,

$$N_{-k} \cup N_{-k+1} \cup \ldots \cup N_0 \cup \ldots \cup N_k = \{n \geq k | f_G(n) = \aleph_0\}.$$

Hence N_l is infinite for some l. We may assume that $l \geq 0$ (by interchanging the roles of G and H, if necessary). By Zippin's Theorem there exist countable groups G_1 and H_1 such that

$$f_{G_1}(n) = \begin{cases} \aleph_0 & \text{if } n \in N_l \\ 0 & \text{if } n \in \omega - N_l \end{cases} \qquad f_{G_1}(a) = f_G(a) \quad \text{for} \quad a \geq \omega;$$

$$f_{H_1}(n) = \begin{cases} \aleph_0 & \text{if } n + l \in N_l \\ 0 & \text{if } n + l \in \omega - N_l, \end{cases} \qquad f_{H_1}(a) = f_H(a) \quad \text{for} \quad a \geq \omega.$$

Let G_2 and H_2 be direct sums of cyclic groups such that

$$f_{G_2}(n) = f_G(n) \quad \text{for} \quad n < \omega,$$
$$f_{H_2}(n) = f_H(n) \quad \text{for} \quad n < \omega.$$

Then we have by the definition of N_l,

(i) $f_{G_1 \oplus G_2}(\xi) = f_{G_1}(\xi) + f_{G_2}(\xi) = f_G(\xi)$ for all ξ,

(ii) $f_{H_1 \oplus H_2}(\xi) = f_{H_1}(\xi) + f_{H_2}(\xi) = f_H(\xi)$ for all ξ,

(iii) $f_{p^l G_1}(\xi) = f_{H_1}(\xi)$ for all ξ.

Therefore, by Ulm's Theorem,

$$G \cong G_1 \oplus G_2, \quad H \cong H_1 \oplus H_2, \quad p^l G_1 \cong H_1,$$

which completes the proof of the Lemma.

2.7. LEMMA. *Let G and H be countable p-groups which are direct sums of cyclic groups which satisfy*

(i) $f_G(n) \leq f_H(n)$ *for all* n;

(ii) $f_H(n)$ *is either* 0 *or* $\aleph_0$ *for all* n;

(iii) *there exists an* $M \geq 0$ *such that if* $f_H(n) = \aleph_0$ *for some* $n \geq M$, *then there is an integer* $m \geq 0$ *for which* $f_G(m) = \aleph_0$ *and* $n - M \leq m \leq n + M$.

Then $G \mathrel{\dot{\cong}} H$.

PROOF. We may assume that $f_G(n) = \aleph_0$ for infinitely many n. Otherwise it follows from (iii) that both G and H are bounded, and hence $G \mathrel{\dot{\cong}} H$. Let $M_0 = \min\{n \geq M \mid f_G(n) = \aleph_0\}$. Let G_1 be a direct sum of cyclic groups such that $f_{G_1}(n) = 0$ for $0 \leq n < M_0$, $f_{G_1}(n) = f_G(n)$ for $n \geq M_0$. Let H_1 be a direct sum of cyclic groups such that $f_{H_1}(n) = 0$ for $0 \leq n < M_0$, $f_{H_1}(n) = f_H(n)$ for $n \geq M_0$. Then $G \mathrel{\dot{\cong}} G_1$, $H \mathrel{\dot{\cong}} H_1$, and G_1 and H_1 satisfy (i), (ii), and (iii). Further, let H_2 be a direct sum of cyclic groups such that $f_{H_2}(n) = f_{H_1}(n + M)$ for all n. Then $H_2 \cong p^M H_1$, so that $H_2 \mathrel{\dot{\cong}} H_1$. Moreover, by (iii), if $f_{H_2}(n) = \aleph_0$, then there exists m with $n \leq m \leq n + 2M$ such that $f_{G_1}(m) = \aleph_0$. By induction, there exist sequences $n_0, n_1, n_2, \ldots$ and $m_0, m_1, m_2, \ldots$ of non-negative integers such that

1) $n_0 \leq m_0 < n_1 \leq m_1 < n_2 \leq m_2 < \ldots$;

2) $m_i \leq n_i + 2M$;

3) $f_{H_2}(n_i) = \aleph_0, f_{G_1}(m_i) = \aleph_0$; and

4) if $m_i < j < n_{i+1}$ or if $j < n_0$, then $f_{H_2}(j) = 0$.

Let $n_0 = M_0 - M$, $m_0 = M_0$. Assuming that $n_0, m_0, n_1, m_1, \ldots, n_i, m_i$ have been defined, let

$$n_{i+1} = \min\{n > m_i \mid f_{H_2}(n) = \aleph_0\}, \quad m_{i+1} = \min\{m \geq n_{i+1} \mid f_{G_1}(m) = \aleph_0\}.$$

The sequences defined in this way satisfy (1), (2), (3), and (4). Let K be a direct sum of cyclic groups such that $f_K(m_i + M) = \aleph_0$, $f_K(n) = 0$ for $n \neq m_i + M$. We will prove that $K \mathrel{\dot{\cong}} G_1$ and $K \mathrel{\dot{\cong}} H_2$. This will complete the proof since $G \mathrel{\dot{\cong}} G_1 \mathrel{\dot{\cong}} K \mathrel{\dot{\cong}} H_2 \mathrel{\dot{\cong}} H_1 \mathrel{\dot{\cong}} H$.

Write

$$K = \sum_{i<\omega} \oplus K_i,$$

where K_i is a direct sum of $\aleph_0$ copies of $Z(p^{m_i+M+1})$. Write

$$K_i = \Big(\sum_{0 \leq j \leq m_i - n_i} \oplus K_{ij}\Big) \oplus K_{im_i},$$

where K_{ij} is a direct sum of $f_{G_1}(n_i + M + j)$ copies of $Z(p^{m_i+M+1})$ and K_{im_i} is a direct sum of $f_{G_1}(m_i) = \aleph_0$ copies of $Z(p^{m_i+M+1})$. If $m_i + M <$

$< j < n_{i+1} + M$, then

$$m_i < j - M < n_{i+1}, \quad \text{so that} \quad f_{G_1}(j) \leq f_{H_1}(j) = f_{H_2}(j - M) = 0$$

by (4). Therefore,

$$G_1 = \sum_{i<\omega} \sum_{n_i+M\leq j\leq m_i+M} \oplus G_{1j},$$

where G_{1j} is a direct sum of $f_{G_1}(j)$ copies of $Z(p^{j+1})$. Moreover,

$$G_1 \simeq \sum_{i<\omega} [(\sum_{n_i+M\leq j\leq m_i+M} \oplus G_{1j}) \oplus G_{1m_i}].$$

Since $m_i - n_i \leq 2M$, it follows that

$$(\sum_{n_i+M\leq j\leq m_i+M} \oplus G_{1j}) \oplus G_{1m_i}$$

is isomorphic to a subgroup L_i of K_i such that

$$p^{2M} K_i \subseteq L_i.$$

Thus, G_1 is isomorphic to a subgroup L of K such that $p^{2M}K \subseteq L$. That is, $G_1 \doteq K$.

Similarly, it is possible to write

$$K_i = \sum_{0\leq j\leq m_i-n_i} \oplus K_{ij}, \quad H_i = \sum_{i<\omega} \sum_{n_i\leq j\leq m_i} \oplus H_{2j},$$

where K_{ij} is a direct sum of $f_{H_2}(n_i + j)$ copies of $Z(p^{m_i+M+1})$ and H_{2j} is a direct sum of $f_{H_2}(j)$ copies of $Z(p^{j+1})$. Obviously H_2 is isomorphic to a subgroup L of K such that $p^{3M} K \subseteq L$. Therefore $H_2 \doteq K$. This completes the proof of the lemma.

2.8. Corollary. *Let G and H be countable p-groups which are direct sums of cyclic groups. Suppose that there exists $M \geq 0$ with the property that if $f_G(n) \neq f_H(n)$ for $n \geq M$, then*

1) *there exists m such that $n - M \leq m \leq n + M$ and $f_G(m) = \aleph_0$;*

2) *there exists m' such that $n - M \leq m' \leq n + M$ and $f_H(m') = \aleph_0$.*

Then $G \doteq H$.

Proof. Write $G = G_1 \oplus G_2$, $H = H_1 \oplus H_2$ where

$$f_{G_1}(n) = \begin{cases} f_G(n) & \text{if } f_G(n) = f_H(n) < \aleph_0 \\ 0 & \text{otherwise,} \end{cases}$$

$$f_{G_2}(n) = \begin{cases} f_G(n) & \text{if } f_G(n) = f_H(n) = \aleph_0 \text{ or } f_G(n) \neq f_H(n) \\ 0 & \text{otherwise,} \end{cases}$$

$$f_{H_1}(n) = \begin{cases} f_H(n) & \text{if } f_G(n) = f_H(n) < \aleph_0 \\ 0 & \text{otherwise,} \end{cases}$$

$$f_{H_2}(n) = \begin{cases} f_H(n) & \text{if } f_G(n) = f_H(n) = \aleph_0 \text{ or } f_G(n) \neq f_H(n) \\ 0 & \text{otherwise.} \end{cases}$$

Then $G_1 \cong H_1$ since $f_{G_1}(n) = f_{H_1}(n)$ for all n. Let K be a direct sum of cyclic groups such that

$$f_K(n) = \begin{cases} 0 & \text{if } f_{G_2}(n) = f_{H_2}(n) = 0 \\ \aleph_0 & \text{otherwise.} \end{cases}$$

Then G_2 and K satisfy (i) and (ii) of Lemma 2.7. Suppose that $f_K(n) = \aleph_0$ for $n \geq M$. Either $f_{G_2}(n) = \aleph_0$ or $f_{G_2}(n) < \aleph_0$. If $f_{G_2}(n) < \aleph_0$, then either $f_{G_2}(n) = f_{H_2}(n) = 0$ or $f_G(n) \neq f_H(n)$. In the first case, $f_K(n) = 0$, which is a contradiction. In the second case, by (1), there exists m such that $n - M \leq m \leq n + M$ and $f_G(m) = \aleph_0$. Therefore $f_{G_2}(m) = \aleph_0$. Hence, G_2 and K also satisfy (iii) of Lemma 2.7, so that $K \doteq G_2$. Similarly, $K \doteq H_2$. Therefore $G_2 \doteq H_2$ and $G \doteq H$.

2.9. Lemma. *Let $\{f_G(n)\}_{n<\omega}$ and $\{f_H(n)\}_{n<\omega}$ be sequences of cardinal numbers which satisfy* (I) *(with $k \geq 1$). Suppose that $b > a + 2k$ and $f_G(j)$ and $f_H(j)$ are finite for $a \leq j \leq b$. Define $\triangle_n$ as in Lemma 2.2 for $a \leq n - k < < n + k \leq b$. Let c, for $a \leq c - k < c + k \leq b$, be such that $\triangle_c \leq \triangle_n$ for all n with $a \leq n - k < n + k \leq b$. Define $g_G^c(m)$ and $g_H^c(m)$ as in Lemma 2.2 for $1 \leq m \leq \sum_{j=c}^{b} f_G(j)$ and $1 \leq m \leq \sum_{j=c}^{b} f_H(j)$, respectively. If $1 \leq m \leq \sum_{j=a}^{c-1} f_G(j)$, define*

$$h_G^c(m) = \max\{r \leq c - 1 \,|\, m \leq \sum_{j=r}^{c-1} f_G(j)\}.$$

Define $h_H^c(m)$ similarly. Then

$$g_H^c(m) - 6k \leq g_G^c(m) \leq g_H^c(m) + 6k$$

for $1 \leq m \leq \min\{\sum_{j=c}^{b-2k-1} f_H(j), \sum_{j=c}^{b} f_G(j)\}$, *and*

$$h_H^c(m) - 6k \leq h_G^c(m) \leq h_H^c(m) + 6k$$

for $1 \leq m \leq \min\{\sum_{j=a+2k+1}^{c-1} f_H(j), \sum_{j=a}^{c-1} f_G(j)\}$.

Proof. By the proof of (III), Lemma 2.2, we have

$$g_H^h(m - t_c) - k \leq g_G^c(m) \tag{1}$$

for $t_c < m \leq \min\{\sum_{j=c-k}^{b} f_H(j), \sum_{j=c}^{b} f_G(j)\}$ and

$$g_G^c(m) \leq g_H^c(m + s_c) + k \tag{2}$$

for $1 \leq m \leq \min\{\sum_{j=c+k}^{b} f_H(j), \sum_{j=c}^{b} f_G(j)\}$.

In a similar way it can be shown that

$$h_H^c(m - s_c) + k \geq h_G^c(m) \tag{3}$$

for $s_c < m \leq \min\{\sum_{j=a}^{c+k-1} f_H(j), \sum_{j=a}^{c-1} f_G(j)\}$ and

$$h_G^c(m) \geq h_H^c(m + t_c) - k \tag{4}$$

for $1 \leq m \leq \min\{\sum_{j=a}^{c-k-1} f_H(j), \sum_{j=a}^{c-1} f_G(j)\}$.

By the proof of (IV), Lemma 2.2, we have

$$g_H^c(n) \leq g_H^c(m) + 2k + 2 \tag{5}$$

for $1 \leq m \leq n \leq \sum_{j=c}^{b} f_H(j)$ and $n - m \leq \triangle_c$.

In a similar way, it follows that if $1 \leq m \leq n \leq \sum_{j=a}^{c-1} f_H(j)$ and $n - m \leq \triangle_c$, then

$$h_H^c(m) \leq h_H^c(n) + 2k + 2. \tag{6}$$

Combining the inequalities (1), (2), and (5) as in Lemma 2.2, we obtain

$$g_H^c(m) - (4k + 2) \leq g_G^c(m) \leq g_H^c(m) + (3k + 2)$$

for $1 \leq m \leq \min\{\sum_{j=c+k}^{b} f_H(j), \sum_{j=c}^{b} f_G(j)\}$. Since $k \geq 1$, we have

$$g_H^c(m) - 6k \leq g_G^c(m) \leq g_H^c(m) + 6k \tag{7}$$

for m in the range given above. It is easy to check, using the minimality of $\triangle_c$, that

$$\sum_{j=c}^{b-2k-1} f_H(j) \leq \sum_{j=c+k}^{b} f_H(j).$$

Hence, (7) holds for $1 \leq m \leq \min\{\sum_{j=c}^{b-2k-1} f_H(j), \sum_{j=c}^{b} f_G(j)\}$.

Using the inequalities (3), (4), and (6), the proof that

$$h_H^c(m) - 6k \leq h_G^c(m) \leq h_H^c(m) + 6k$$

for $1 \leq m \leq \min\{\sum_{j=a+2k+1}^{c-1} f_H(j), \sum_{j=a}^{c-1} f_G(j)\}$ is similar.

2.10. PROOF THAT (I) AND (II) ARE SUFFICIENT FOR THE QUASI-ISOMORPHISM OF COUNTABLE p-GROUPS. Suppose that G and H satisfy (I) and (II). If $f_G(n)$ and $f_H(n)$ are almost all finite then $G \doteq H$ by Lemma 2.4. Therefore, we may assume that there are infinitely many n such that $f_G(n) = \aleph_0$

or $f_H(n) = \aleph_0$. By Lemma 2.5 there are infinitely many n with $f_G(n) = \aleph_0$ and infinitely many n with $f_H(n) = \aleph_0$. By Lemma 2.6 we may assume that G and H are direct sums of cyclic groups. Further, we may suppose that $f_G(n) = \aleph_0$ if and only if $f_H(n) = \aleph_0$. Indeed, let G' be the direct sum of cyclic groups such that $f_{G'}(n) = f_G(n)$ if $f_G(n)$ and $f_H(n)$ are both finite, and $f_{G'}(n) = \aleph_0$ if either $f_G(n)$ or $f_H(n)$ is infinite. Similarly, let H' be the direct sum of cyclic groups such that $f_{H'}(n) = f_H(n)$ if $f_G(n)$ and $f_H(n)$ are both finite, and $f_{H'}(n) = \aleph_0$ if either $f_G(n)$ or $f_H(n)$ is infinite. The groups G' and H' satisfy (I) and have the property that $f_{G'}(n) = \aleph_0$ if and only if $f_{H'}(n) = \aleph_0$. If $f_G(n) \neq f_{G'}(n)$ for $n \geq k$, then $f_H(n) = \aleph_0$, and by Lemma 2.5, $f_G(m) = \aleph_0$ with $n - k \leq m \leq n + k$. Moreover, $f_{G'}(n) = \aleph_0$. Thus, by Corollary 2.8, $G \doteq G'$. Similarly, $H \doteq H'$.

Let $0 \leq d_1 < d_2 < d_3 < \ldots$ be all of the integers for which $f_G(d_i) = f_H(d_i) = \aleph_0$. It follows from Corollary 2.8 that we can assume that either $d_{i+1} = d_i + 1$, or else $d_{i+1} > d_i + 2k$. To prove this, let G' be a direct sum of cyclic groups such that

$$f_{G'}(n) = \begin{cases} \aleph_0 & \text{if } n \leq d_1 \text{ and } d_1 \leq 2k \\ \aleph_0 & \text{if } d_i \leq n \leq d_{i+1} \text{ and } d_{i+1} - d_i \leq 2k \\ f_G(n) & \text{otherwise.} \end{cases}$$

Suppose that $f_G(n) \neq f_{G'}(n)$ for $n \geq 2k$. Then $f_{G'}(n) = \aleph_0$, and $f_G(m) = \aleph_0$ with $n - 2k \leq m \leq n + 2k$. By Corollary 2.8, $G \doteq G'$. Similarly, if H' is a direct sum of cyclic groups such that

$$f_{H'}(n) = \begin{cases} \aleph_0 & \text{if } n \leq d_1 \text{ and } d_1 \leq 2k \\ \aleph_0 & \text{if } d_i \leq n \leq d_{i+1} \text{ and } d_{i+1} - d_i \leq 2k \\ f_H(n) & \text{otherwise,} \end{cases}$$

then $H \doteq H'$. Moreover, G' and H' satisfy (I).

Let the complement of the set $d_1, d_2, \ldots$ be written as a union of disjoint (maximal) intervals:

$$I_1 = [a_1, b_1],\ I_2 = [a_2, b_2],\ \ldots,$$

where $0 \leq a_1 < b_1 < a_2 < b_2 < \ldots, b_i > a_i + 2k$, $f_G(a_i - 1) = f_H(a_i - 1) = \aleph_0$ for $i > 1$, and $f_G(b_i + 1) = f_H(b_i + 1) = \aleph_0$ for $i \geq 1$. Write

$$G = \Big(\sum_{i\geq 1} \oplus A_i\Big) \oplus \Big(\sum_{i\geq 1} \oplus C_i\Big),$$

$$H = \Big(\sum_{i\geq 1} \oplus B_i\Big) \oplus \Big(\sum_{i\geq 1} \oplus D_i\Big),$$

where A_i and B_i are direct sums of $\aleph_0$ cyclic groups of order p^{d_i+1},

$$C_i = \sum_{j\in I_i} \oplus \Big(\sum_{f_G(j)} \oplus Z\,(p^{j+1})\Big),$$

$$D_i = \sum_{j\in I_i} \oplus \Big(\sum_{f_H(j)} \oplus Z\,(p^{j+1})\Big).$$

For each i, choose $c_i \in [a_i, b_i]$, for $a_i \leq c_i - k < c_i + k \leq b_i$, so that $\triangle_{c_i} \leq \triangle_{n_i}$ for all n_i with $a_i \leq n_i - k < n_i + k \leq b_i$. Moreover, from among those c_i for which $\triangle_{c_i}$ is minimal, select one such that

$$\sum_{j=c_i-k}^{c_i+k} f_G(j) \text{ is minimal.}$$

Then we can write

$$C_i = \Big(\sum_{m=1}^{e_i} \oplus\{x_{im}\}\Big) \oplus \Big(\sum_{m=1}^{f_i} \oplus\{y_{im}\}\Big),$$

$$D_i = \Big(\sum_{m=1}^{g_i} \oplus\{z_{im}\}\Big) \oplus \Big(\sum_{m=1}^{h_i} \oplus\{w_{im}\}\Big),$$

where

$$e_i = \sum_{j=c_i}^{b_i} f_G(j), \quad f_i = \sum_{j=a_i}^{c_i-1} f_G(j), \quad g_i = \sum_{j=c_i}^{b_i} f_H(j), \quad h_i = \sum_{j=a}^{c_i-1} f_H(j),$$

$$E(x_{im}) = g_G^{c_i}(m) + 1, \quad E(y_{im}) = h_G^{c_i}(m) + 1, \quad E(z_{im}) = g_H^{c_i}(m) + 1,$$
$$E(w_{im}) = h_H^{c_i}(m) + 1.$$

Let $u_i = \min\Big\{e_i, \sum_{j=c_i}^{b_i-2k-1} f_H(j)\Big\}$, $v_i = \min\Big\{f_i, \sum_{j=a_i+2k+1}^{c_i-1} f_H(j)\Big\}$. Write

$$G' = \Big(\sum_{i\geq 1} \oplus A_i\Big) \oplus \Big(\sum_{i\geq 1} \oplus \Big(\Big(\sum_{m=u_i+1}^{e_i} \oplus\{x_{im}\}\Big) \oplus \Big(\sum_{m=v_i+1}^{f_i} \oplus\{y_{im}\}\Big)\Big)\Big),$$

$$H' = \Big(\sum_{i\geq 1} \oplus B_i\Big) \oplus \Big(\sum_{i\geq 1} \oplus \Big(\Big(\sum_{m=u_i+1}^{g_i} \oplus\{z_{im}\}\Big) \oplus \Big(\sum_{m=v_i+1}^{h_i} \oplus\{w_{im}\}\Big)\Big)\Big),$$

$$C'_i = \Big(\sum_{m=1}^{u_i} \oplus \{x_{im}\}\Big) \oplus \Big(\sum_{m=1}^{v_i} \oplus \{y_{im}\}\Big),$$

$$D'_i = \Big(\sum_{m=1}^{u_i} \oplus \{z_{im}\}\Big) \oplus \Big(\sum_{m=1}^{v_i} \oplus \{w_{im}\}\Big).$$

Then

$$G = G' \oplus \sum_{i\geq i} \oplus C'_i, \quad H = H' \oplus \sum_{i\geq 1} \oplus D'_i.$$

By Lemma 2.9,

$$|E(x_{im}) - E(z_{im})| \leq 6k,$$

for $1 \leq m \leq u_i$, and

$$|E(y_{im}) - E(w_{im})| \leq 6k$$

for $1 \leq m \leq v_i$. Therefore, there exists a group K_i which is a direct sum of cyclic groups containing subgroups C''_i, D''_i such that

$$C'_i \cong C''_i, \quad D'_i \cong D''_i, \quad p^{6k} K_i \subseteq C''_i, \quad p^{6k} K_i \subseteq D''_i.$$

Thus, there is a group $K = \sum_{i\geq 1} \oplus K_i$ such that

$$\sum_{i\geq 1} \oplus C'_i \mathrel{\dot{\cong}} K \mathrel{\dot{\cong}} \sum_{i\geq 1} \oplus D'_i .$$

It remains to show that $G' \mathrel{\dot{\cong}} H'$. This will follow from Corollary 2.8 after the following results are established.

(1) $$E(x_{im}) \geq b_i - 11k \text{ for } u_i < m \leq e_i.$$

Suppose that $E(x_{im}) < b_i - 11k$. Then

$$b_i - 11k > E(x_{im}) = g_G^{c_i}(m) + 1 \geq g_G^{c_i}(u_i) + 1 \geq g_H^{c_i}(u_i) - 6k + 1 .$$

Hence $g_H^{c_i}(u_i) < b_i - 5k$. Therefore

$$\sum_{j=c_i}^{b_i-5k-1} f_H(j) \geq u_i = \sum_{j=c_i}^{b_i-2k-1} f_H(j) .$$

Consequently, $f_H(j) = 0$ for $b_i - 5k \leq j \leq b_i - 2k - 1$. Thus,

$$\triangle_{c_i} = \sum_{c_i-k}^{c_i+k} f_H(j) \leq \sum_{b_i-5k}^{b_i-3k} f_H(j) = 0$$

Hence $f_H(j) = 0$ for $c_i - k \leq j \leq c_i + k$. By (I),

$$\sum_{j=c_i}^{b_i-3k-1} f_G(j) \leq \sum_{j=c_i-k}^{b_i-2k-1} f_H(j) = \sum_{j=c_i}^{b_i-2k-1} f_H(j) = u_i < m.$$

Therefore, $g_H^{c_i}(m) \geq b_i - 3k$. Hence

$$b_i - 11k > g_G^{c_i}(m) + 1 \geq b_i - 3k + 1 ,$$

which is a contradiction.

(2) $$E(z_{im}) \geq b_i - 11k \text{ for } u_i < m \leq g_i.$$

Assume that $E(z_{im}) < b_i - 11k$. There are two cases to consider. First, suppose that $u_i = \sum_{j=c_i}^{b_i-2k-1} f_H(j) \leq \sum_{j=c_i}^{b_i} f_G(j)$. Since $m > u_i$, it follows that $g_H^{c_i}(m) \geq b_i - 2k$. Also, $b_i - 11k > E(z_{im}) = g_H^{c_i}(m) + 1$. Therefore,

$$b_i - 11k > g_H^{c_i}(m) + 1 \geq b_i - 2k + 1 ,$$

which is a contradiction. We now suppose that $u_i = e_i = \sum_{j=c_i}^{b_i} f_G(j) \leq \sum_{j=c_i}^{b_i-2k-1} f_H(j)$. We have

$$b_i - 11k > E(z_{im}) = g_H^{c_i}(m) + 1 \geq g_H^{c_i}(u_i) + 1 \geq g_G^{c_i}(u_i) - 6k .$$

Since $g_G^{c_i}(u_i) < b_i - 5k$, it follows that

$$\sum_{j=c_i}^{b_i-5k-1} f_G(j) \geq u_i = \sum_{j=c_i}^{b_i} f_G(j) .$$

Hence $f_G(j) = 0$ for $b_i - 5k \leq j \leq b_i$. By (I),

$$\sum_{j=b_i-3k}^{b_i-k} f_H(j) \leq \sum_{j=b_i-4k}^{b_i} f_G(j) = 0 .$$

Therefore, by the choice of c_i,

$$\sum_{j=c_i-k}^{c_i+k} f_G(j) \leq \sum_{j=b_i-3k}^{b_i-k} f_G(j) = 0 .$$

Hence $f_G(j) = 0$ for $c_i - k \leq j \leq c_i + k$. By (I),

$$\sum_{j=c_i}^{b_i-k} f_H(j) \leq \sum_{j=c_i-k}^{b_i} f_G(j) = \sum_{j=\ i}^{b_i} f_G(j) = u_i < m .$$

Thus, $g_H^{c_i}(m) > b_i - k$. We have

$$b_i - 11k > g_H^{c_i}(m) + 1 > b_i - k + 1 ,$$

which is a contradiction.

(3) $$E(y_{im}) \leq a_i + 11k \text{ for } v_i < m \leq f_i .$$

(4) $$E(w_{im}) \leq a_i + 11k \text{ for } v_i < m \leq h_i .$$

The proofs of (3) and (4) are similar to those of (1) and (2). It follows from (1), (2), (3), and (4) that there exists a bound M such that if $f_{G'}(n) \neq \neq f_{H'}(n)$ then $f_{G'}(m) = \aleph_0$ and $f_{H'}(m') = \aleph_0$ with $n - M \leq m \leq n + M$ and $n - M \leq m' \leq n + M$. By Corollary 2.8, $G' \doteq H'$. This completes the proof.

Reference

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MÉTHODES TOPOLOGIQUES EN THÉORIE DES GROUPES ABÉLIENS

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Tous les groupes considérés sont abéliens. Nous désignons par Z l'ensemble des nombres entiers et par Q l'ensemble des nombres rationnels. Pour le reste nous suivons les notations et la terminologie de Fuchs [2].

(A) METHODES TOPOLOGIQUES

Considérations générales

D'une façon précise mais peut-être un peu déraisonnable on peut proposer comme but de la théorie des groupes abéliens la recherche et l'étude de tous les foncteurs qu'on peut définir sur la catégorie des groupes abéliens ou sur une sous-catégorie des groupes abéliens. Notre esprit est trop faible pour connaître de façon complète les catégories usuelles en mathématique et en général seule une petite partie d'une catégorie nous est connue et familière. Un foncteur permet de projeter sur la catégorie de départ un peu de la lumière que nous avons sur la catégorie d'arrivée.

Le point de vue que nous venons de développer est justifié par le fait que la plus grande part de la théorie des groupes abéliens peut s'exprimer en langage de foncteurs. Cela ne veut pas dire qu'il est souhaitable de le faire, car il serait difficile d'échapper à beaucoup de lourdeur et de pédantisme. Mais il est certain que la considération de l'aspect fonctoriel d'une théorie permet de mieux comprendre et classer les problèmes qu'on peut se poser. En théorie des groupes abéliens on recherche des théorèmes de structures pour des classes de groupes aussi générales que possible. C'est presque toujours à partir de l'étude d'un foncteur qu'on obtient un théorème de structure.

Signalons encore avec un peu de perversité un avantage tout-à-fait contingent du point de vue que nous venons d'exposer: il permet de faire rentrer à peu près toutes les mathématiques dans le cadre proposé par L. Fuchs à ce colloque, pour lui conserver des dimensions raisonnables.

Exemples de foncteurs en théorie des groupes abéliens

Désignons par $\mathcal{G}$ la catégorie des groupes abéliens et par $\mathcal{G}(E)$ la catégorie des groupes abéliens munis d'une filtration de type E, où E est un ensemble ordonné. (On dit qu'un groupe G est muni d'une filtration de type E si on a défini sur G une famille de sous-groupes $(G_i)_{i \in E}$ telle que $i \leq j$ entraîne $G_i \subseteq G_j$ (filtration croissante) ou $G_i \supseteq G_j$ (filtration décroissante). Les morphismes de $\mathcal{G}(E)$ sont les morphismes u: $G \to H$ tels que $u(G_i) \subseteq H_i$ pour tout $i \in E$.)

Un cas fréquent de foncteur F: $\mathcal{G} \to \mathcal{G}(E)$ est celui où on a $F(G) = G$ pour tout groupe et $F(u) = u$ pour tout homomorphisme. Pour que F soit un foncteur il faut et il suffit que pour tout homomorphisme u: $G \to H$ et tout $i \in E$ on ait $u(G_i) \subseteq H_i$. Il est clair que ceci nécessite qu'on ait des filtrations par des sous-groupes *complètement invariants*. L'exemple le plus familier est celui de la filtration p-adique définie par $G_n = p^n G$. On peut faire varier n dans l'ensemble des entiers ≥ 0 ou dans l'ensemble des ordinaux.

Un exemple plus élaboré est ce qu'on pourrait appeler le foncteur socle. A p premier on associe le foncteur F_p qui à G fait correspondre son socle $P = G[p]$ muni de la filtration des $P_n = P \cap p^n G$, le morphisme $F_p(u)$ étant défini de façon naturelle. Si on fait varier n dans l'ensemble des ordinaux la donnée du socle de G comme groupe filtré détermine les invariants d'Ulm de G. On sait que beaucoup de propriétés des groupes primaires peuvent être démontrées de façon élégante en «remontant» à partir du socle ce qui est typiquement dans l'esprit de la théorie des foncteurs.

Méthodes topologiques en théorie des groupes abéliens

On peut appeler méthodes topologiques l'utilisation de foncteurs définis sur la catégorie des groupes abéliens ou une sous-catégorie et à valeurs dans la catégorie $\overline{\mathcal{G}}$ des groupes abéliens topologiques. Si F est un tel foncteur la condition essentielle est donc que $F(u)$ soit un homomorphisme *continu* pour tout homomorphisme u.

Un cas fréquent de foncteur F: $\mathcal{G} \to \overline{\mathcal{G}}$ est celui où on a $F(G) = G$ pour tout groupe et $F(u) = u$ pour tout homomorphisme (cela revient à munir chaque groupe d'une topologie compatible avec la structure de groupe). Pour que F soit un foncteur il faut et il suffit que *tout homomorphisme u soit continu*. Dans ce cas nous dirons qu'on a défini sur la catégorie des groupes abéliens ou la sous-catégorie considérée une *topologie fonctorielle*. Un exemple familier est celui qui consiste à munir tout groupe de la topologie p-adique, ou encore de la topologie Z-adique (obtenue en prenant les nG, $n \in Z$ comme système fondamental de voisinages de 0).

Un autre exemple consiste à associer à un entier premier p le foncteur F_p qui à G fait correspondre son socle $P = G[p]$ muni de la topologie induite par la topologie p-adique de G.

Voici un exemple de foncteur $\mathcal{G} \to \overline{\mathcal{G}}$ que nous recontrerons dans la suite.

Définition. Etant donné un groupe Γ on appelle *Γ-topologie* sur un groupe G la topologie qui admet comme système fondamental de voisinages de 0 les intersections finies de noyaux d'homomorphismes de G dans Γ.

Vérifions que cette topologie est fonctorielle, c'est-à-dire que tout homomorphisme u: $G \to H$ est continu lorsque G et H sont munis des Γ-topologies. Soient $u_1, \ldots, u_n$ des homomorphismes de H dans Γ, on a:

$$\overset{-1}{u}\left(\bigcap_{i=1}^{n} \operatorname{Ker}(u_i)\right) = \bigcap_{i=1}^{n} \overset{-1}{u}\left(\operatorname{Ker}(u_i)\right) = \bigcap_{i=1}^{n} \operatorname{Ker}(u_i u)$$

où $u_1 u, \ldots, u_n u$ sont des homomorphismes de G dans Γ.

Remarque 1. On peut définir la Γ-topologie sur G comme la topologie la moins fine rendant continues toutes les applications $u \in \operatorname{Hom}(G, \Gamma)$ lorsqu'on munit Γ de la topologie discrète.

Remarque 2. A. Kertész et T. Szele [6] ont montré qu'on peut toujours mettre sur un groupe abélien infini une topologie séparée compatible avec la structure de groupe. Il serait intéressant de reprendre cette question en imposant à la topologie d'être fonctorielle et d'étudier de façon générale les topologies fonctorielles.

Nous allons maintenant faire quelques applications de méthodes topologiques. Comme d'habitude quand on utilise des méthodes topologiques en algèbre, cela veut dire qu'on fait beaucoup d'algèbre mélangée avec un peu de topologie.

(B) ETUDE DU FONCTEUR $G \to \operatorname{Hom}(G, \Gamma)$

Dans tout ce paragraphe nous considérons uniquement des groupes *sans torsion*. Etant donné un groupe G nous notons $H_p(x)$ la hauteur d'un élément x de G relativement à l'entier premier p et $T(x)$ son type (se reporter à [2] pour la définition de ces notions). Si G est de rang 1 son type $T(G)$ est le type commun de tous ses éléments non nuls. Etant donné deux types R et S définis respectivement par les suites (r_p) et (s_p) nous définissons $R + S$ par la suite $(r_p + s_p)$.

Dans tout ce qui suit Γ est un groupe sans torsion de rang 1. Nous dirons que $G^* = \operatorname{Hom}(G, \Gamma)$ est le dual de G relativement à Γ. Nous allons étudier les relations entre G et G^* ce qui nous montrera dans quelle mesure il est acceptable de parler de dualité entre G et G^*.

Etant donné $x^* \in G^*$ nous posons $\langle x, x^* \rangle = x^*(x)$. Si dans $\langle x, x^* \rangle$ on fixe x et fait varier x^* on obtient un *homomorphisme canonique* φ de G dans G^{**}. Pour que φ soit injectif il faut et il suffit que pour tout élément $x \neq 0$ de G il existe $x^* \in G^*$ tel que $\langle x, x^* \rangle \neq 0$.

Relations d'orthogonalité entre G et G^*

On dit que $x \in G$ et $x^* \in G^*$ sont *orthogonaux* si $\langle x, x^* \rangle = 0$. Etant donné $A \subseteq G$ on note $A^\perp$ l'ensemble des $x^* \in G^*$ qui sont orthogonaux à tous les $x \in A$. Etant donné $A \subseteq G^*$ on note $A^\perp$ l'ensemble des $x \in G$ qui sont orthogonaux à tous les $x^* \in A$. On a les propriétés élémentaires suivantes:

$$A^\perp \text{ est un sous-groupe pur,}$$

$$A \subseteq B \Longrightarrow A^\perp \supseteq B^\perp, \qquad A^{\perp\perp} \supset A,$$

$$(A \cup B)^\perp = A^\perp \cap B^\perp, \qquad (A \cap B)^\perp \supseteq A^\perp + B^\perp.$$

THÉORÈME 1. *Si $x_1^*, \ldots, x_n^*$ sont des éléments de G^* l'ensemble $\{x_1^*, \ldots, x_n^*\}^{\perp\perp}$ est le sous-groupe pur de G^* engendré par $x_1^*, \ldots, x_n^*$.*

$x^* \in \{x_1^*, \ldots, x_n^*\}^{\perp\perp}$ signifie $\operatorname{Ker}(x^*) \supseteq \bigcap_{i=1}^{n} \operatorname{Ker}(x_i^*)$. Nous allons en déduire qu'il existe des entiers $r \neq 0$ et $r_1, \ldots, r_n$ tels que $r\,x^* = \sum_{i=1}^{n} r_i\, x_i^*$. Pour $n = 0$ la propriété est triviale donc on peut raisonner par écurrence sur n. Si $\operatorname{Ker}(x^*) \supseteq \bigcap_{i=1}^{n-1} \operatorname{Ker}(x_i^*)$ la propriété est démontrée. Sinon il existe $x \in \bigcap_{i=1}^{n-1} \operatorname{Ker}(x_i^*)$ tel que $\langle x, x_n^* \rangle \neq 0$. Comme Γ est de rang 1 il existe des entiers $s \neq 0$ et t tels que $\langle x, sx^* + tx_n^* \rangle = 0$. L'homomorphisme $s\,x^* + t\,x_n^*$ s'annule sur $\bigcap_{i=1}^{n-1} \operatorname{Ker}(x_i^*)$ car il s'annule sur $\bigcap_{i=1}^{n} \operatorname{Ker}(x_i^*)$ et sur $\bigcap_{i=1}^{n-1} \operatorname{Ker}(x_i^*) \,/\, \bigcap_{i=1}^{n} \operatorname{Ker}(x_i^*)$. Il existe donc des entiers $m \neq 0$ et $r_1, \ldots, r_{n-1}$ tels que $m\,(s\,x^* + t\,x_n^*) = \sum_{i=1}^{n-1} r_i x_i^*$ ce qui achève la démonstration.

Topologies faibles sur G et G^*

La topologie faible sur G est la topologie $\sigma(G, G^*)$ qui admet comme système fondamental de voisinages de 0 les ensembles $A^\perp$ où A parcourt les parties finies de G^*. La topologie $\sigma(G, G^*)$ est donc ce que nous avons appelé plus haut la Γ-topologie sur G.

La topologie faible sur G^* est la topologie $\sigma(G^*, G)$ qui admet comme système fondamental de voisinages de 0 les ensembles $A^\perp$ où A parcourt les parties finies de G. En général $\sigma(G^*, G)$ est strictement moins fine que la Γ-topologie.

Si on munit Γ de la topologie discrète (qui coïncide d'ailleurs avec la Γ-topologie) et G de la topologie $\sigma(G, G^*)$ tout homomorphisme $x^* \in G^*$ est continu. Le dual topologique de G coïncide donc avec G^*; nous noterons G' ce dual topologique *muni de la topologie* $\sigma(G^*, G)$. Il est intéressant de déterminer le dual topologique G'' de G'.

THÉORÈME 2. *Γ étant muni de la topologie discrète et G^* de la topologie $\sigma(G^*, G)$ l'ensemble des $y \in G^{**}$ qui sont continus est le sous-groupe pur de G^{**} engendré par $\varphi(G)$.*

Dire que $y \in G^{**}$ est continu signifie que y s'annule sur un voisinage de 0 dans G^*. Un tel voisinage de 0 peut être pris de la forme $\{x_1, \ldots, x_n\}^\perp = \{\varphi(x_1), \ldots, \varphi(x_n)\}^\perp$. En appliquant le théorème 1 à G^{**} on voit que y appartient au sous-groupe pur engendré par $\varphi(x_1), \ldots, \varphi(x_n)$ ce qui démontre le théorème.

Nous pouvons maintenant préciser dans quelle mesure on peut parler de dualité. D'un point de vue strict on peut parler de dualité algébrique si φ est bijectif (ce qui permet d'identifier G et G^{**}) et de dualité faible si φ est injectif et $\varphi(G)$ pur dans G^{**} (ce qui permet d'identifier G et G''). Bien que ces conditions ne soient pas en général réalisées on a les propriétés suivantes qui montrent qu'il est acceptable d'utiliser le langage de la dualité:

(a) Le sous-groupe pur de G'' engendré par $\varphi(G)$ est G''.

(b) $(G'')^*$ est canoniquement isomorphe à G^*.

(c) Sur G^* les topologies $\sigma(G^*, G)$ et $\sigma(G^*, G'')$ coïncident.

(a) Résulte du théorème 2. (b) résulte de ce que tout homomorphisme $x^* \in G^*$ opère canoniquement sur G^{**} et de (a). Enfin (c) résulte de ce que $A^\perp$ n'est pas changé lorsqu'on remplace $A \subseteq G$ par le sous-groupe pur de G^{**} engendré par $\varphi(A)$.

REMARQUE 1. La condition $G'' = G^{**}$ est nécessaire et suffisante pour que $\sigma(G^*, G)$ coïncide avec la Γ-topologie.

REMARQUE 2. La topologie $\sigma(G^*, G)$ que nous avons introduite sur G^* est la topologie de la convergence simple sur le groupe Hom (G, Γ), Γ étant muni de la topologie discrète. La topologie de la convergence simple sur les groupes d'homomorphismes a déjà été utilisée en théorie des groupes abéliens, en particulier par T. Szele [9].

Transposé d'un homomorphisme $u: G \to H$

Le transposé de u est l'homomorphisme $u^*: H^* \to G^*$ défini par:

$$\langle x, u^*(y^*)\rangle = \langle u(x), y^*\rangle \qquad x \in G, \quad y^* \in H^*.$$

On a $(u+v)^* = u^* + v^*$ et $(u\,v)^* = v^*u^*$ chaque fois que ces égalités ont un sens. u est continu lorsqu'on munit G et H des topologies faibles puisque celles-ci coïncident avec la Γ-topologie.

Théorème 3. *u^* est un homomorphisme continu de H^* dans G^* lorsqu'on munit H^* de la topologie $\sigma(H^*, H)$ et G^* de la topologie $\sigma(G^*, G)$.*

Soit V un voisinage de 0 dans G^*, qu'on peut prendre de la forme $\{x_1, \ldots, x_n\}^\perp$ où $x_1, \ldots, x_n \in G$. Son image réciproque par u^* est $\{u(x_1), \ldots, u(x_n)\}^\perp$ qui est un voisinage de 0 dans H^*.

Remarque. Ce théorème montre qu'on a un foncteur à valeurs dans les groupes abéliens topologiques lorsqu'on pose $G \to G'$ et $u \to u^*$.

Comparaison de u et u^{**}. Soit φ (resp. ψ) l'application canonique de G (resp. H) dans G^{**} (resp. H^{**}). On voit que $u^{**}\varphi$ est un prolongement de $\psi\,u$ et que $u^{**}(G'') \subseteq H''$.

Etude du cas où G est de rang 1

Théorème 4. *Soit G un groupe sans torsion de rang 1. Si $T(G) \leq T(\Gamma)$ le groupe G^* est de rang 1 et $T(G^*)$ est le plus grand type tel que $T(G) + T(G^*) = T(\Gamma)$; de plus φ est injectif et on a pour tout $x \in G$ et tout p premier la relation $H_p(\varphi(x)) = H_p(x)$ si p ne divise pas Γ et $H_p(\varphi(x)) = \infty$ si p divise Γ. Si $T(G)$ n'est pas $\leq T(\Gamma)$ on a $G^* = 0$.*

Il est clair que G^* est au plus de rang 1 et que $x^* \in G^*$ est déterminé par la connaissance de $\langle x, x^*\rangle$ pour un élément $x \neq 0$ de G. On peut choisir pour $\langle x, x^*\rangle$ tout élément de Γ vérifiant:

$$H_p(\langle x, x^*\rangle) \geq H_p(x) \qquad \forall\ p \text{ premier}$$

Le choix de $\langle x, x^*\rangle \neq 0$ est possible si et seulement si $T(G) \leq T(\Gamma)$. Le type de G^* s'obtient en calculant le type d'un élément non nul x^* de G^*:

$$H_p(x^*) = \begin{cases} H_p(\langle x, x^*\rangle) - H_p(x) & \text{si } H_p(x) \text{ est fini} \\ \infty & \text{si } H_p(x) = \infty \end{cases}$$

Ceci montre que $T(G^*)$ est le plus grand type tel que $T(G) + T(G^*) = T(\Gamma)$.

De $G^* \neq 0$ résulte φ injectif puisque G étant de rang 1 on a $\langle x, x^*\rangle \neq 0$ pour tout $x \neq 0$ dès que $x^* \neq 0$. On a :

$$H_p(\langle x, x^*\rangle) = H_p(x) + H_p(x^*)$$

$$H_p(\langle x^*, \varphi(x)\rangle) = H_p(x^*) + H_p(\varphi(x)).$$

Comme φ est défini par $\langle x^*, \varphi(x)\rangle = \langle x, x^*\rangle$ on en déduit $H_p(x) = H_p(\varphi(x))$, sous réserve que $H_p(\langle x, x^*\rangle) \neq \infty$. Si $H_p(\langle x, x^*\rangle) = \infty$ c'est-à-dire si p divise Γ, alors p divise G^{**}.

Corollaire. *Si $T(G) \leq T(\Gamma)$ et si tout p premier qui divise Γ divise G l'homomorphisme canonique φ est bijectif.*

C'est en particulier le cas si on prend $\Gamma = G$.

Dualité entre somme directe et produit direct

Si $G = \sum_{i\in\Lambda} G_i$ on a $G^* \cong \prod_{i\in\Lambda} G_i^*$ mais si $G = \prod_{i\in\Lambda} G_i$ on peut seulement affirmer que $G^* \supseteq \sum_{i\in\Lambda} G_i^*$. L'égalité $G^* = \sum_{i\in\Lambda} G_i^*$ équivaut au fait que la Γ-topologie sur G est le produit des Γ-topologies sur les G_i. Pour préciser des cas dans lesquels cette égalité a lieu nous aurons besoin des propriétés et notions suivantes:

Définition. Un groupe L sans torsion est dit *maigre* (slender) si tout homomorphisme u d'un produit d'une infinité dénombrable de groupes cycliques infinis $\{a_i\}$ dans L est tel que $u(a_i) = 0$ pour presque tout i (c'est-à-dire pour tous les i à l'exception d'un nombre fini).

On a le théorème suivant pour la démonstration duquel nous renvoyons à [7] (Dans [2] seul un résultat plus faible est démontré):

Théorème 5. (Sąsiada [7]). *Un groupe sans torsion dénombrable réduit est maigre.*

On a encore le théorème suivant pour la démonstration duquel nous renvoyons à [2]:

Théorème 6. (Łoś) *Soit $G = \prod_{i\in\Lambda} G_i$ où les G_i sont des groupes sans torsion. Un groupe maigre L possède les propriétés suivantes relativement à G:*

(i) *Si u est un homomorphisme de G dans L on a $\varphi(G_i) = 0$ pour presque tout i.*

(ii) *Si u est un homomorphisme de G dans L tel que $\varphi(G_i) = 0$ pour tout i et si de plus $|\Lambda|$ est un cardinal de mesure nulle alors $u = 0$.*

Revenons maintenant à l'étude de la Γ-dualité.

Théorème 7. *Γ étant un groupe de rang 1 distinct du groupe additif des rationnels soient $G = \prod_{i\in\Lambda} A_i$ et $H = \sum_{i\in\Lambda} B_i$ où $|\Lambda|$ est un cardinal de mesure nulle, $A_i \cong \Gamma$, $B_i \cong \mathrm{Hom}(\Gamma, \Gamma)$. Alors on a $\mathrm{Hom}(G, \Gamma) \cong H$ et $\mathrm{Hom}(H, \Gamma) \cong G$.*

On a $H^* = \prod_{i\in\Lambda} B_i^*$ et du théorème 4 résulte $B_i^* \cong A_i$. Des théorèmes 5 et 6 résulte $G^* = \sum_{i\in\Lambda} A_i$, d'où la Γ-dualité annoncée entre G et H.

Remarque. $\Gamma^* = \mathrm{Hom}(\Gamma, \Gamma)$ est le groupe additif des nombres rationnels dont le dénominateur divise Γ. On peut faire opérer l'anneau

Γ^* de ces nombres rationnels sur Γ et Γ^* donc sur G et H; H devient alors un Γ^*-module *libre*.

Forme explicite de la Γ-dualité entre G et H. Les hypothèses étant celles du théorème 7 un élément $x \in G$ peut être représenté par une famille $(x_i)_{i\in\Lambda}$ où $x_i \in \Gamma$ et un élément $x^* \in H$ peut être représenté par une famille $(x^i)_{i\in\Lambda}$ où $x^i \in \Gamma^*$ est nul pour presque tout i. La Γ-dualité entre G et H peut alors être définie par la forme bilinéaire:

$$\langle x, x^* \rangle = \sum_{i\in\Lambda} \langle x_i, x^i \rangle .$$

L'anneau des endomorphismes de G. Les hypothèses étant celles du théorème 7 on voit par transposition que l'anneau $E(G)$ des endomorphismes de G est inversement isomorphe à l'anneau $E(H)$ des endomorphismes de H.

Soit $M_\Lambda(E(\Gamma))$ l'anneau des matrices (u_i^j) où $i, j \in \Lambda$ et $u_i^j \in E(\Gamma)$ avec la restriction $u_i^j = 0$ pour presque tout j lorsque i est fixé. On peut identifier $E(H)$ avec $M_\Lambda(E(\Gamma))$ en associant à $u = (u_i^j)$ l'endomorphisme défini par les équations:

$$y^i = \sum_{j\in\Lambda} u_j^i x^j \qquad (i \in \Lambda) .$$

Le transposé u^* de u est alors défini par les équations:

$$y_i = \sum_{j\in\Lambda} u_i^j x_j \qquad (i \in \Lambda) .$$

Nous dirons que (u_i^j) est la matrice de u^*.

Décomposition de G en produit. Les hypothèses étant celles du théorème 7 on peut chercher toutes les décompositions de G sous la forme $G = \prod_{i\in\Lambda} G_i$ où $G_i \cong \Gamma$. Il résulte du théorème 7 qu'elles correspondent aux décompositions de H sous la forme $H = \sum_{i\in\Lambda} H_i$ où $H_i \cong \Gamma^* = \mathrm{Hom}(\Gamma, \Gamma)$. Avec la forme explicite donnée plus haut de la Γ-dualité entre G et H on a les relations:

$$H_i = (\prod_{j\neq i} G_j)^\perp, \qquad G_i = (\sum_{j\neq i} H_j)^\perp .$$

Remarque. Dans le cas où $\Gamma = Z$ la plupart des résultats qui précèdent figurent dans [1]. Dans le cas où de plus Λ est dénombrable les propriétés $H \cong \mathrm{Hom}(G, Z)$ et $G \cong \mathrm{Hom}(H, Z)$ figurent dans Specker [8]. Il y a lieu aussi de signaler l'article de E. C. Zeeman [10], d'après lequel, toujours dans le cas $\Gamma = Z$, le théorème 7 serait vrai sans restriction sur $|\Lambda|$ (moyennant un axiome d'accessibilité).

Le problème 23 de L. Fuchs [2]: *Soit G un produit de groupes cycliques infinis. Trouver des conditions nécessaires et suffisantes sur une famille $(a_i)_{i\in\Lambda}$ d'éléments de G pour que l'on ait $G \cong \prod_{i\in\Lambda} \{a^i\}$.*

Si G est de la forme $\prod_{i\in\Lambda} A_i$ où $A_i \cong Z$ et où $|\Lambda|$ est de mesure nulle on peut donner une réponse à ce problème. Soit $(a^i_j)_{i\in\Lambda}$ les coordonnées de a^i relativement à la décomposition $G \cong \prod_{i\in\Lambda} A_i$. Si l'on suppose que $G \cong \prod_{i\in\Lambda} \{a^i\}$ il existe un automorphisme u de G tel que $u(e^i) = a^i$ où e^i est l'élément de G de coordonnées $(\delta^j_i)_{j\in\Lambda}$ relativement à la décomposition $G = \prod_{i\in\Lambda} A_i$. La matrice de l'automorphisme u étant (a^i_j) nous pouvons énoncer: Pour que $G \cong \prod_{i\in\Lambda} \{a_i\}$ il faut et il suffit que la matrice (a^i_j) appartienne à $M_\Lambda(Z)$ et soit inversible.

Remarques sur le cas général

Nous ne savons pas si la partie du théorème 4 relative à $H_p(\varphi(x))$ est vraie dans le cas général. Si on reprend les calculs faits pour un couple x, x^* tel que $\langle x, x^*\rangle \neq 0$ on obtient en considérant les homomorphismes $x^*\colon G/\mathrm{Ker}\,(x^*) \to \Gamma$ et $\varphi(x)\colon G^*/\mathrm{Ker}\,(\varphi(x)) \to \Gamma$:

$$H_p(\langle x, x^*\rangle) = H_p(x + \mathrm{Ker}\,(x^*)) + H_p(x^*),$$

$$H_p(\langle x^*, \varphi(x)\rangle) = H_p(\varphi(x)) + H_p(x^* + \mathrm{Ker}\,\varphi(x)).$$

Si $H_p(\langle x, x^*\rangle) \neq \infty$ on en déduit seulement:

$$H_p(\varphi(x)) + H_p(x^* + \mathrm{Ker}\,\varphi(x)) = H_p(x + \mathrm{Ker}\,(x^*)) + H_p(x^*).$$

Si $H_p(\langle x, x^*\rangle) = \infty$ il en résulte $H_p(\varphi(x)) = \infty$. Ceci montre que si $\Gamma = Q$ le groupe G'' est l'enveloppe divisible de G.

Nous avons dans tout ce qui précède supposé que Γ était de rang 1. Cependant il est intéressant de remarquer que les résultats qui concernent la dualité entre somme et produit peuvent être généralisés à partir d'un système de trois groupes A, B, Γ tels que

$$\mathrm{Hom}\,(A, \Gamma) \cong B, \qquad \mathrm{Hom}\,(B, \Gamma) \cong A.$$

Si Γ est maigre le théorème 7 sera valable pour des groupes $G = \prod_{i\in\Lambda} A_i$ où $A_i \cong A$ et $H = \sum_{i\in\Lambda} B_i$ où $B_i \cong B$.

Ceci nous conduit au problème suivant: Etant donné un groupe A trouver un groupe Γ maigre et un groupe B (aussi simple que possible) tels que $\mathrm{Hom}\,(A, \Gamma) \cong B$ et $\mathrm{Hom}\,(B, \Gamma) \cong A$.

Dans le cas d'un groupe A tel que $\mathrm{Hom}\,(A, A) \cong Z$ on a comme solution évidente $B = A$ et $\Gamma = Z$.

(C) QUESTIONS DIVERSES

Sous-groupes de base et topologie p-adique

La notion de sous-groupe de base a été introduite par L. Kulikov pour les p-groupes. Elle a été généralisée comme suit par L. Fuchs [3] pour les groupes quelconques: On dit que B est un p-sous-groupe de base de G si

(i) B est une somme directe de groupes cycliques dont l'ordre est une puissance de p ou infini,

(ii) B est p-pur dans G,

(iii) le groupe quotient G/B est divisible par p.

La condition (iii) est équivalente à la condition 'B dense dans G pour la topologie p-adique'. En effet B dense dans G pour la topologie p-adique signifie $B + p^n G = G$ pour tout entier n, or $B + p^n G = G$ exprime que G/B est divisible par p^n.

Dans le cas où G est un p-groupe rien n'est à changer dans la définition d'un p-sous-groupe de base si ce n'est qu'on peut remplacer (i) par 'B est une somme directe de groupes cycliques' (qui seront nécessairement des p-groupes). L'interprétation topologique de la condition (iii) donnée plus haut est due à L. Kaloujnine [4].

L'interprétation topologique de la condition (iii) dans le cas d'un groupe quelconque permet d'utiliser les théorèmes sur les applications continues dans les espaces uniformes. Soit u un homomorphisme de B dans un groupe H. Si H est complet et séparé pour la topologie p-adique u se prolonge d'une façon et d'une seule à un homomorphisme de G dans H.

Remarque. Si G est un p-groupe on peut dans ce qui précède supposer seulement que H est un sous-groupe p-fermé au sens de L. Kulikov, c'est-à-dire coïncide avec le sous-groupe de torsion de son complété p-adique $\hat{H}$: Tout homomorphisme de G dans H se prolonge à un homomorphisme u de G dans $\hat{H}$ mais on a nécessairement $u(G) \subseteq H$. D'ailleurs on peut éviter de faire intervenir le complété p-adique de H en remplaçant la topologie p-adique sur H par la limite inductive des topologies induites par la topologie p-adique sur les $H[p^n]$. Dire que H est p-fermé équivaut en effet à dire que H est complet pour cette nouvelle topologie. Bien entendu il faut aussi munir G de cette nouvelle topologie et vérifier son caractère fonctoriel, ce qui est facile.

Puissance de l'ensemble des sous-groupes de base d'un p-groupe

Le théorème suivant donne une réponse complète au problème 8 de L. Fuchs [2]:

Théorème 8. (*S. Khabbaz et E. A. Walker*). *Soient G un p-groupe, D sa partie divisible, B un sous-groupe de base G et $b(G)$ la puissance de l'ensemble des sous-groupes de base de G.*

(i) *Si G est borné ou divisible on a $b(G) = 1$.*

(ii) *Si $G = D + B$, D de rang fini m, $B = B_1 + \ldots + B_k$ où $B_i = Z(p_i^n)$ on a $b(G) = \prod_{i=1}^{k} p^{mn_i}$.*

(iii) *Dans tous les autres cas on a $b(G) = |G|^{|B|}$.*

La démonstration se fait en distinguant un certain nombre de cas. Notre but est de donner une démonstration par une méthode topologique lorsque G est réduit et non borné. On sait dans ce cas que $|G| \leqq 2^{|B|}$. Par ailleurs $b(G)$ ne peut dépasser $|G|^{|B|} \leqq (2^{|B|})^{|B|} = 2^{|B|}$ donc tout revient a montrer que $b(G) = 2^{|B|}$. Il suffit pour cela de démontrer que $b(B) = 2^{|B|}$.

Soit P le socle de B muni de la filtration des $P_n = P \cap p^n B$ et de la topologie associée. Pour qu'un sous-groupe R de P soit le socle d'un sous-groupe de base de G il faut et il suffit que R soit dense dans P. En effet dire que R est dense dans P signifie que $R + P_n = P$ quel que soit n. Nous allons maintenant considérer P comme espace vectoriel sur le corps discret $Z/(p)$, P étant toujours muni de la topologie définie par les P_n. Le théorème dans le cas actuellement envisagé sera une conséquence du lemme suivant:

Lemme. *La puissance de l'ensemble des sous-espaces de P qui sont denses dans P et de codimension 1 est $2^{|P|}$.*

Tout sous-espace de codimension 1 de P est défini par une forme linéaire $f \neq 0$ du dual algébrique P^* de P. On sait que la dimension et la puissance de P^* sont égales à $2^{|P|}$. On sait d'autre part que si $f \in P^*$ le noyau de f est un sous-espace dense dans P ou fermé. Pour que f soit continue il faut et il suffit que le noyau de f soit fermé. Donc tout revient à démontrer que l'ensemble des $f \in P^*$ qui sont discontinues a pour puissance $2^{|P|}$, c'est-à-dire finalement que l'ensemble P' des formes linéaires continues est distinct de P^*.

$f \in P^*$ est continue si et seulement si f s'annule sur un sous-espace P_n. Il en résulte:

$$P' = \sum_{n=0}^{\infty} (P_n/P_{n+1})^* \subset \prod_{n=0}^{\infty} (P_n/P_{n+1})^* = P^*$$

ce qui achève la démonstration du lemme.

Conditions pour qu'un sous-groupe soit facteur direct

Si H est facteur direct d'un groupe G alors H est le noyau de l'homomorphisme qui consiste à projeter G sur le cofacteur de H. Nous pouvons donc énoncer:

THÉORÈME 9. *Pour qu'un sous-groupe H d'un groupe G soit facteur direct il est nécessaire qu'il soit fermé pour toute topologie fonctorielle séparée qu'on peut définir sur G.*

Comme application considérons un groupe de torsion T non borné et séparé pour la topologie Z-adique. Soit $\hat{T}$ le complété de T pour la topologie Z-adique et H, G des groupes tels que $T \subseteq H \subset G \subseteq \hat{T}$ et G pur dans $\hat{T}$. Le groupe H ne peut pas être facteur direct de G puisqu'il est dense dans G pour la topologie Z-adique. On notera que $\hat{T}$ contient toujours des éléments d'ordre infini et que si les composantes primaires de T sont bornées T est le sous-groupe de torsion de $\hat{T}$.

Voici dans un ordre d'idées voisin un théorème inspiré d'un article de A. Kertész et T. Szele [5]:

THÉORÈME 10. *Soit G un groupe abélien tel que pour tout p premier on ait une décomposition $G = r_p G \oplus H_p$ où $r_p = p^{n(p)}$ avec $n(p) \geq 1$. Alors on a une décomposition $G = D \oplus H$ où D est divisible et où H possède les propriétés suivantes :*

(i) *H_p est la composante primaire de H associée à p et $r_p H_p = 0$,*

(ii) *$\sum_p H_p$ est dense dans H pour la topologie Z-adique.*

Remarquons d'abord que $r_p H_p \subseteq H_p \cap r_p G = 0$. Il en résulte la propriété $r_p G = r_p^2 G$ qui sera utile plus loin. Le théorème revient à démontrer que $D = \bigcap_p r_p G$ est divisible. En effet si D est divisible on peut trouver une décomposition $G = D \oplus H$ avec $H \supseteq \sum_p H_p$. Le fait que $\sum_p H_p$ est dense dans H résulte de $H \cong G/D$ et de l'existence d'une injection canonique de G/D dans $\prod_p (G/r_p G_p) \cong \prod_p H_p$.

Tout revient donc à démontrer que $D = \bigcap_p r_p G$ est divisible c'est-à-dire $q D = D$ pour tout q premier. Ce sera une conséquence du lemme:

LEMME. *Soient (A_i) une famille de sous-groupes de G et u un endomorphisme de G. Si pour tout $i \neq 1$ on a* $\operatorname{Ker}(u) \subseteq A_i$ *alors $u(\bigcap_i A_i) = \bigcap_i u(A_i)$.*

Il suffit de démontrer que $u(\bigcap_i A_i) \supseteq \bigcap_i u(A_i)$. Si $x \in \bigcap_i u(A_i)$ il existe pour tout i un $a_i \in A_i$ tel que $x = u(a_i)$. Si $i \neq 1$ de $u(a_1) = u(a_i)$ résulte $a_1 - a_i \in \operatorname{Ker}(u) \subseteq A_i$ d'où $a_1 \in A_i$. On a alors $a_1 \in \bigcap_i A_i$ d'où $x \in u(\bigcap_i A_i)$.

En appliquant le lemme à $D = \bigcap_p r_p G$ et $u = q$ ce qui est permis puisque Ker $(u) = G[q] \subseteq r_p G$ pour tout $p \neq q$, on obtient:

$$q D = \bigcap_p q r_p G = \bigcap_p (q G \cap r_p G) = (q G) \cap D = D .$$

(On a utilisé $q r_p G = q G \cap r_p G$: C'est toujours vrai pour $p \neq q$ et c'est vrai dans le cas présent pour $p = q$ en vertu de $p r_p G \supseteq r_p^2 G = r_p G$ et $p G \supseteq r_p G$).

En appliquant le théorème 10 au cas $r_p = p$ on obtient le résultat suivant:

COROLLAIRE. (*Kertész et Szele* [5]). *Une condition nécessaire et suffisante pour que tout multiple de G soit facteur direct est que $G = D + H$ où D est divisible et où H vérifie les conditions suivantes:*

(i) *Si H_p est la composante primaire de H associée à p on a $pH_p = 0$.*
(ii) *$\sum_p H_p$ est dense dans H pour la topologie Z-adique.*

(La suffisance des conditions énoncées ne résulte pas du théorème 10 mais est facile à démontrer).

A. Kertész et T. Szele ont posé le problème plus difficile de trouver tous les groupes G dont toute image endomorphe est facteur direct. Si on suppose G réduit alors les composantes primaires de G sont telles que $p G_p = 0$ et $\sum_p G_p$ dense dans G pour la topologie Z-adique. Ces conditions ne sont pas suffisantes pour que toute image endomorphe de G soit facteur direct, comme le montre l'exemple suivant:

Soit pour chaque p premier un groupe $T_p \neq 0$ tel que $p T_p = 0$. Formons $G = H \oplus K$ où $H = \sum_p T_p$ et $K = \prod_p T_p$. On a un endomorphisme u de G en posant $u(K) = 0$ et en envoyant H dans K de façon naturelle. L'adhérence de $u(G)$ pour la topologie Z-adique est K, donc $u(G)$ n'est pas fermé pour la topologie Z-adique, donc $u(G)$ n'est pas facteur direct.

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ON A CONJECTURE OF PIERCE CONCERNING DIRECT DECOMPOSITIONS OF ABELIAN GROUPS

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At the recent Colloquium on Abelian Groups held at Tihany in Hungary R. S. Pierce announced his conjecture that there exists an Abelian p-group that is isomorphic with the direct sum of three copies of itself, but not with the direct sum of two copies of itself, and suggested that a theorem of mine might be used to settle the corresponding question for torsion-free groups.[1] In this paper I shall show that there does exist a torsion-free group with Pierce's property; this fact is the case $r = 2$ of the following more general result.

THEOREM. *Let r be a positive integer. Then there is a countable reduced torsion-free Abelian group G with the property that for positive integers m, n the isomorphism*

$$\sum_m G \cong \sum_n G \tag{1}$$

holds if and only if $m \equiv n \pmod{r}$. In particular,

$$G \cong \sum_{r+1} G , \tag{2}$$

while for every integer s such that $1 < s < r + 1$

$$G \not\cong \sum_s G .$$

($\sum_m G$ denotes the direct sum of m copies of the Abelian group G.)

Three corollaries of the theorem are given at the end of the paper.

The proof of the theorem makes use of my Theorem A [1], which we quote here as

[1] W. G. Leavitt has in effect established the truth of the corresponding conjecture (and several generalizations) for modules over certain specially constructed rings [6]. For a discussion of the analogous problems for Boolean algebras, see Halmos [3].

LEMMA 1. *Let* $\mathbf{A}$ *be a ring whose additive group is countable, reduced, and torsion-free.*[2] *Then* $\mathbf{A}$ *is isomorphic with the endomorphism ring* $\mathbf{E}(G)$ *of some countable, reduced, torsion-free Abelian group* G.

We shall also need the following criterion for the isomorphism of two direct summands of an Abelian group in terms of its endomorphism ring.

LEMMA 2. *Let* ω_1, ω_2 *be two idempotents in the endomorphism ring* $\mathbf{E}(G)$ *of an Abelian group* G. *Then the direct summands* $G\,\omega_1$, $G\,\omega_2$ *are isomorphic if and only if there exist elements* $x, y \in \mathbf{E}(G)$ *such that*

$$xy = \omega_1, \quad yx = \omega_2. \tag{3}$$

PROOF. Suppose that $x, y \in \mathbf{E}(G)$ satisfy (3). Then the identities

$$xyx = x\,\omega_2 = \omega_1\,x, \qquad yxy = y\,\omega_1 = \omega_2\,y$$

show that the endomorphisms xyx and yxy of G induce homomorphisms $x^* : G\,\omega_1 \to G\,\omega_2$ and $y^* : G\,\omega_2 \to G\,\omega_1$, respectively; and it follows from the further identities

$$(xyx)\,(yxy) = \omega_1, \qquad (yxy)\,(xyx) = \omega_2$$

that x^* and y^* are inverse isomorphisms between $G\,\omega_1$ and $G\,\omega_2$. Conversely, if $x^* : G\,\omega_1 \to G\,\omega_2$ and $y^* : G\,\omega_2 \to G\,\omega_1$ are inverse isomorphisms, then the endomorphisms $x = \omega_1\,x^*$, $y = \omega_2\,y^*$ of G are solutions of (3).

PROOF *of the theorem*. Take $\mathbf{A}$ to be the ring freely generated by symbols α_i, β_i $(i = 1, 2, \ldots, r+1)$ subject to the relations

$$\tag{4} \begin{gathered} \beta_i\,\alpha_j = \begin{cases} 1 & (i = j), \\ 0 & (i \neq j), \end{cases} \\ \sum_{i=1}^{r+1} \alpha_i\,\beta_i = 1. \end{gathered}$$

Suppose first that we have proved (a) that the additive group of $\mathbf{A}$ is free Abelian of countable rank, and (b) that there exists a function T: $\mathbf{A} \to Z/rZ$ such that

(i) $T(x+y) = T(x) + T(y)$ for all $x, y \in \mathbf{A}$;

(ii) $T(xy) = T(yx)$ for all $x, y \in \mathbf{A}$;

(iii) $T(1) = 1_r$,

where 1_r denotes the residue class of 1 modulo r. Then by (a) and Lemma 1 there is a countable reduced torsion-free Abelian group G whose endomorphism ring $\mathbf{E}(G)$ is isomorphic with $\mathbf{A}$. We make the identification $\mathbf{E}(G) = \mathbf{A}$, and write

$$\omega_i = \alpha_i\,\beta_i \;(i = 1, 2, \ldots, r+1).$$

[2] All rings in this paper are associative and have identity elements.

It follows at once from the relations (4) that the ω_i are orthogonal idempotents with the sum 1, so that we have the direct decomposition

$$G = \sum_{i=1}^{r+1} G\omega_i .$$

Moreover, by Lemma 2 the equations $\beta_i \, a_i = 1$, $a_i \, \beta_i = \omega_i$ imply that the $G \, \omega_i$ are all isomorphic with G itself. Thus we have established the isomorphism (2); and it follows immediately that the isomorphism (1) holds whenever $m \equiv n \pmod r$. Now suppose for a contradiction that an isomorphism (1) holds for some pair of positive integers m, n such that $m \not\equiv n \pmod r$. By virtue of what we have just proved it is clear that an isomorphism (1) must also hold for some pair m, n with $1 \leq m < n \leq r$, so that

$$\sum_{i=1}^{m} G\omega_i \cong \sum_{i=1}^{n} G\omega_i .$$

According to Lemma 2 **A** must therefore contain elements x, y such that

$$xy = \sum_{i=1}^{m} \omega_i , \qquad yx = \sum_{i=1}^{n} \omega_i .$$

Taking 'traces' we deduce that

$$\sum_{i=1}^{m} T(\omega_i) = T(xy) = T(yx) = \sum_{i=1}^{n} T(\omega_i) .$$

But $T(\omega_i) = T(a_i \, \beta_i) = T(\beta_i \, a_i) = T(1) = 1_r$; so this equation in traces is equivalent to the congruence $m \equiv n \pmod r$, and we have the required contradiction.

The theorem will therefore be proved once we have established (a) and (b).

For the proof of (a) and (b) it is convenient to identify **A** in the obvious way with the quotient $\mathbf{B}/\mathfrak{A}$, where **B** is the ring freely generated by symbols a_i, b_i $(i = 1, 2, \ldots, r+1)$ subject to the relations

$$b_i \, a_i = \begin{cases} 1 & (i = j) , \\ 0 & (i \neq i) , \end{cases} \tag{5}$$

and $\mathfrak{A}$ is the ideal of **B** generated by the single element

$$z = 1 - \sum_{i=1}^{r+1} a_i \, b_i . \tag{6}$$

Standard arguments show that the additive group of **B** is free Abelian on the distinct non-vanishing products in the a_i, b_i;[3] and the identities (5)

[3] See, for example, the theory of monomial rings in Rédei [7].

imply that every such non-vanishing product is equal to a product of the form

$$(7) \qquad a_{i_1} a_{i_2} \ldots a_{i_m} b_{j_n} \ldots b_{j_2} b_{j_1},$$

where $m, n \geq 0$ and $1 \leq i_k, j_k \leq r + 1$. Hence the products (7) constitute a free basis of the additive group of $\mathbf{B}$.

To prove (a) it is only necessary to show that $\mathfrak{A}$ is a direct summand of the additive group of $\mathbf{B}$. Now it is clear that $\mathfrak{A}$ is additively generated by the elements $w_1 z w_2$, where w_1, w_2 are products of the form (7); and since (5), (6) imply that $b_i z = 0 = z a_i$ $(i = 1, 2, \ldots, r + 1)$, $\mathfrak{A}$ is even additively generated by the elements of the restricted form

$$(8) \qquad u = (a_{i_1} a_{i_2} \ldots a_{i_m}) \left(1 - \sum_{i=1}^{r+1} a_i b_i\right) (b_{j_n} \ldots b_{j_2} b_{j_1}),$$

where $m, n \geq 0$ and $1 \leq i_k, j_k \leq r + 1$. An entirely straightforward Steinitz Exchange argument, whose details are left to the reader, shows that the system of all elements of the form (7) remains a basis of the additive group of $\mathbf{B}$ when the subsystem of all elements for which $m, n \geq 1$ and $i_m = j_n = r + 1$ is replaced by the system of all elements of the form (8). We conclude that $\mathfrak{A}$ is a direct summand of the additive group of $\mathbf{B}$; and (a) follows, as we have already observed.

To prove (b) it is clearly enough to construct a function $T^*\colon \mathbf{B} \to Z/rZ$ that satisfies the analogues (i*), (ii*), (iii*) of (i), (ii), (iii) and, in addition, vanishes on every element u of the form (8). We define T^* on the basis elements (7) by the rule,

$$T^*(a_{i_1} \ldots a_{i_m} b_{j_n} \ldots b_{j_1}) = \begin{cases} 1 & \text{if } m = n \text{ and } i_k = j_k \ (1 \leq k \leq m), \\ 0 & \text{otherwise}, \end{cases}$$

and extend T^* to the whole of $\mathbf{B}$ by additivity. By construction T^* certainly satisfies (i*) and (iii*). The vanishing of $T^*(u)$ for every u of the form (8) is also easily verified: if the expression for $T^*(u)$ is expanded with the help of (i*), then the terms of the expansion all vanish except in the case $m = n$, $i_k = j_k$ $(1 \leq k \leq m)$, when the expansion reduces to $1_r - (r + 1)1_r$, which vanishes in Z/rZ. Finally, in view of (i*), in order to verify (ii*) it is enough to consider the case in which x is one of the additive generators (7) and y is one of the multiplicative generators a_i, b_i and in this case the verification is trivial. Taking for example the case $x = a_{i_1} a_{i_2} \ldots a_{i_m} b_{j_n} \ldots b_{j_2} b_{j_1}$, $y = a_i$, we find that $T^*(xy)$ and $T^*(yx)$ both vanish unless $n = m + 1$ and

$$j_k = \begin{cases} i & (k = 1), \\ i_{k-1} & (k = 2, \ldots, n), \end{cases}$$

when $T^*(xy) = T^*(yx) = 1_r$.

The proof of the theorem is complete.

We end with three applications of the theorem.

Corollary 1. *There exist countable torsion-free Abelian groups* G, H, U *such that* $G \cong H + U$, $H \cong G + U$, *but* $G \not\cong H$.

This provides a negative answer to de Groot's formulation [2] of Kaplansky's First Test Problem [5]. For a related counter-example in the class of countable torsion-free Abelian groups, see Corner [1]. The earliest counter-example to the problem was obtained by Sąsiada [8]. A counter-example in the class of uncountable Abelian p-groups has recently been obtained by Crawley.[4]

Proof. Take the group G of the theorem for $r = 2$, with $H = G + G$, $U = G$.

Corollary 2. *For any positive integer* r *there exist torsion-free Abelian groups* G, H *of countable rank such that* $\sum_r G \cong \sum_r H$, *but* $\sum_s G \not\cong \sum_s H$ *for* $1 \leq s < r$.

The case $r = 2$, which gives a negative answer to Kaplansky's Second Test Problem [5], and the case $r = 3$, have already been obtained by Jónsson [4] and by G. Higman (unpublished), respectively; Jónsson's and Higman's groups are even of finite rank. Crawley's counter-example referred to above implies the existence of non-isomorphic uncountable Abelian p-groups G, H such that $G + G \cong H + H$.

Proof. Take the group G of the theorem for the given r, with $H = G + G$.

Corollary 3. *There is a torsion-free Abelian group* G *of countable rank that is isomorphic with the direct sum of any finite number* ($\neq 0$) *of copies of itself, but not with the direct sum of infinitely many copies of itself.*

Note that the complete direct sum of countable many copies of the integers provides a simple example of a torsion-free Abelian group of *uncountable* rank with the property of Corollary 3.

Proof. Take the group G of the theorem for $r = 1$. Then, according to the theorem, we have for every integer $n \geq 1$

$$G \cong \sum_n G.$$

But G cannot be isomorphic with the direct sum of infinitely many copies of itself; for otherwise $\mathbf{E}(G)$ would be of cardinal of the continuum, whereas we know from the construction that $\mathbf{E}(G)$ is countable.

Note that an example going in the opposite direction to that of Corollary 3 is not possible: any group that is isomorphic with the direct sum of

[4] Crawley's counter-example was mentioned by Professor Pierce at the Tihany Colloquium.

infinitely many copies of itself is necessarily also isomorphic with the direct sum of any finite number ($\neq 0$) of copies of itself.

I should like to thank Professor R. S. Pierce both for communicating his conjecture to me, and also for drawing my attention to the paper of Leavitt [6]. I should like also to express my gratitude to Professor L. Fuchs, the Hungarian Academy of Sciences, and the Bolyai János Mathematical Society for organizing the very successful Tihany Colloquium, and to the International Mathematical Union for a travel grant.

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A GENERALIZATION OF DEPENDENCE RELATIONS

By

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A GA-dependence structure $(\bar{S}, \delta)$ has been defined [1] as a set $\bar{S}$ with a GA-dependence relation $\delta \subseteq \bar{S} \times \mathfrak{P}\,\bar{S}$ ('an element depends on a subset') satisfying certain 6 axioms. The concept of the GA-dependence relation is a generalization of the previously introduced 'algebraic' dependence relations and covers also the dependence in Abelian groups—thus being a solution of the problem indicated by T. Szele [4].

The dependence relations in Abelian groups serve to some extent as a model for this concept. As a consequence, e.g. the δ-rank of a GA-dependence structure (whose invariance can be established through a certain generalization of the Steinitz' Exchange Theorem) has similar properties to those of the ranks of an Abelian group. Certain 'direct' decompositions, compositions and factorizations (preserving 'linearity' of the rank function) can be defined in the theory of GA-dependence structures generalizing the corresponding concepts of the theory of Abelian groups.

In the case of a 'linear' dependence relation (see e.g. [5]) it is possible to present in a simple way an axiomatization in terms of independent subsets, or to establish a lattice representation of the respective structure (see e.g. [6, 3, 2]). From this point of view, the lack of the 'transitive' axiom for a GA-dependence relation makes the study more difficult. Thus, as to the first question, to a given class of independent sets (satisfying certain 3 axioms) there corresponds, in general, a whole family of GA-dependence relations with the least and the greatest elements. The following theorem relates to the other question (for the sake of simplicity the formulation assumes that $\bar{S} = S$, i.e. that the subsets of the singular and the neutral elements are void):

Let (S, δ) be a GA-dependence structure. Then there exists a set L (finite if S is finite and of the same cardinality as S otherwise) and a one-to-one mapping φ of $\mathfrak{P}\,S$ into $\mathfrak{P}\,L$ such that for $x \in S$ and $X, X_1, X_2 \in \mathfrak{P}\,S$

$$X_1 \subseteq X_2 \leftrightarrow \varphi(X_1) \subseteq \varphi(X_2)$$

and

$$[x, X] \in \delta \longleftrightarrow \varphi((x)) \cap \varphi(X) \neq \emptyset .$$

Thus, any such GA-dependence structure can be represented as a complete atomic Boolean algebra of some subsets of a set (with the operations different from set-theoretical ones) in such a way that the set-theoretical intersection represents the dependence relation. The axioms of the GA-dependence structure can be rewritten as certain conditions for such a Boolean algebra. From this, the definition of the GA-dependence structure can be read in a symmetric form in terms of 'a subset depends on a subset'.

The detailed treatment of the subject will appear in the *Czech. Math. Journal*.

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ON THE SUBGROUPS OF AN ABELIAN GROUP THAT ARE IDEALS IN EVERY RING

By

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Let us consider an (additive) Abelian group G. Rings (not necessarily associative) with an additive group isomorphic to G are called rings over G. In Fuchs [1] certain subgroups of G are shown to be ideals in all rings over G. The aim of this note is to characterize, in a group-theoretic way, the subgroups H of G which are ideals in all rings over G.

Let $\mathfrak{E}(G)$ denote the endomorphism ring of G,[1] and $E(G)$ the additive group of $\mathfrak{E}(G)$. Let furthermore $I(G)$ be the subgroup of $E(G)$ generated by all homomorphic images of G into $E(G)$, i.e.

$$I(G) = \{G\eta, \text{ for all } \eta \in \operatorname{Hom}(G, E(G))\}.$$

Then we have

LEMMA 1. *$I(G)$ is the additive group of an ideal $\mathfrak{J}(G)$ of $\mathfrak{E}(G)$.*

PROOF. Let $\eta \in \operatorname{Hom}(G, E(G))$ and $\varphi \in \mathfrak{E}(G)$. The mappings $g \to (g\eta)\varphi$ and $g \to \varphi(g\eta)$ are because of $(g\eta)\varphi, \varphi(g\eta) \in E(G)$, and

$$\left.\begin{aligned} \big((g+h)\eta\big)\varphi &= (g\eta + h\eta)\varphi = (g\eta)\varphi + (h\eta)\varphi \\ \varphi\big((g+h)\eta\big) &= \varphi(g\eta + h\eta) = \varphi(g\eta) + \varphi(h\eta) \end{aligned}\right\} (g, h \in G)$$

homomorphisms of G into $E(G)$. Consequently, $(G\eta)\varphi$ and $\varphi(G\eta)$ are contained in $I(G)$. This shows that the elements of $I(G)$ form an ideal $\mathfrak{J}(G)$ of $\mathfrak{E}(G)$.

THEOREM 1. *A subgroup H of G is an ideal in every ring over G if and only if it is an $\mathfrak{J}(G)$-module.*

PROOF. Let us consider a ring R over G. We associate with each $g(\in G)$ the right-sided multiplication g_R by g in this ring R. By the right-sided distributive law g_R is an endomorphism of G, and because of the left-sided distributive law the mapping η_R: $g \to g_R$ is a homomorphism of G into $E(G)$ (and so into $I(G)$). Thus the elements of a subgroup H of G form a

[1] We shall denote the image of an element $g \in G$ under the endomorphism η by $g\eta$. Accordingly, the product $\eta\chi$ of two endomorphisms η, χ of G is defined as $g(\eta\chi) = (g\eta)\chi$.

right ideal in the ring R if and only if H is a $G\eta_R$-module. (A similar proof applies for the left ideals.) Thus, if H is an $\mathfrak{J}(G)$-module, then H is a (two-sided) ideal in every ring over G.

Now, let H be an ideal in every ring over G. Let further η be a homomorphism of G into $E(G)$. We define the multiplication in G in the following way:

$$g \cdot h = (g)h\eta \qquad (g, h \in G)\,,$$

i.e. we apply $h\eta$ to g. It is obvious that in this way we get a ring over G. Since H is also an ideal in this ring, H is a $G\eta$-module. This holds for all $G\eta$, therefore H must be an $\mathfrak{J}(G)$-module.

Remark. Obviously, a subgroup H of G is a one-sided ideal in all rings over G if and only if it is an $\mathfrak{J}(G)$-module.

The fully invariant subgroups are ideals in all rings over G. Let us consider what other subgroups of G are ideals in all rings over G.

Definition. Let H be a subgroup of G. Put

H^* = *the subgroup of* G *generated by the elements* $g(\in G)$ *such that* $g\varphi \in H$ *holds for each* $\varphi \in \mathfrak{J}(G)$:

$\overline{H}$ = *the subgroup of* G *generated by the elements* $g(\in G)$ *for which there exist an* $h \in H$ *and a* $\varphi \in \mathfrak{J}(G)$ *such that* $g = h\varphi$.

From Lemma 1, and from the definition, we obtain immediately:

Lemma 2. *For a subgroup* H *of* G, *both* H^* *and* $\overline{H}$ *are fully invariant subgroups of* G.

Lemma 3. *The following inclusions hold:*

(i) $\overline{H^*} \subseteqq H \subseteqq \overline{H}^*$ *if* H *is a subgroup of* G;

(ii) $\overline{H}_1 \subseteqq \overline{H}_2$ *and* $H_1^* \subseteqq H_2^*$ *if* H_1, H_2 *are subgroups of* G *such that* $H_1 \subseteqq H_2$.

Now we shall give a characterization of the subgroups in question.

Theorem 2. *For a subgroup* H *of* G, *the following assertions are equivalent:*

(i) H *is an ideal in every ring over* G;

(ii) $\overline{H} \subseteqq H$;

(iii) *there exists a fully invariant subgroup* T *of* G *such that* $T \subseteqq H \subseteqq T^*$;

(iv) *there exists a fully invariant subgroup* S *of* G *such that* $\overline{S} \subseteqq H \subseteqq S$;

(v) $H \subseteqq H^*$.

Proof. The equivalence of (i) and (ii) has been proved in Theorem 1.

To deduce (iii) from (ii), let $\overline{H} \subseteqq H$. Because of Lemma 2, $T = \overline{H}$ is fully invariant in G; and $T \subseteqq H$. Further in view of Lemma 3 (i), we have $H \subseteqq \overline{H}^* = T^*$.

Next we prove that (iii) implies (iv). Let T be a fully invariant subgroup of G, and $T \subseteqq H \subseteqq T^*$. Let $S = T^*$; then S is fully invariant in G; and $H \subseteqq S$. By Lemma 3 (i), $\overline{S} = \overline{T^*} \subseteqq T \subseteqq H$.

Assume (iv). Let S be fully invariant in G, and let $\overline{S} \subseteqq H \subseteqq S$. Because of Lemma 3 (ii), from $\overline{S} \subseteqq H$ we obtain $\overline{S}^* = H^*$. By Lemma 3 (i), we hence have $H \subseteqq S \subseteqq \overline{S}^* \subseteqq H^*$, which establishes (v).

Finally, we prove (ii) from (v). Let $H \subseteqq H^*$. From Lemma 3 (ii) and (i) we conclude that $\overline{H} \subseteqq \overline{H^*} \subseteqq H$. This completes the proof.

REMARK 1. It is easy to see that if, for some H, $\overline{H} \subseteqq H$ holds, then for the fully invariant subgroups T and S, $T \subseteqq H \subseteqq T^*$ and $\overline{S} \subseteqq H \subseteqq S$ hold if and only if $\overline{H} \subseteqq T \subseteqq H$ and $H \subseteqq S \subseteqq H^*$ are satisfied.

REMARK 2. Theorem 2 implies that for a fully invariant subgroup T of G, *all* groups H with $T \subseteqq H \subseteqq T^*$ are ideals in all rings over G.

REMARK 3. It is easy to prove that 0^* is the intersection of all annihilators of the rings over G. (The annihilator of a ring R is the set of all $r \in R$ satisfying $rx = xr = 0$ for all $x \in R$.) Indeed, $g \in 0^*$ means $g\eta = 0$ for all $\eta \in I(G)$, and because $I(G)$ is generated by all homomorphic images of G in $E(G)$, this means $\mu(x, g) = 0 = \mu(g, x)$ for all $x \in G$ and for all multiplications $\mu(x, y)$ over G. Thus $0^* = G$ if and only if G is a nil group according to Fuchs [1]. If G is a p-group, then 0^* is the set of all elements of infinite height in G, i.e. if G is a torsion group, then $0^* = \bigcap_n nG = G^1$ is the first Ulm subgroup of G.

Now we consider the groups in which only the fully invariant subgroups are ideals in all the rings over them. To these groups we shall refer as *I-groups.*

Let $N(g)$ be the set of all endomorphisms of G which annihilate the element $g \in G$. $N(g)$ is obviously a right ideal of $\mathfrak{E}(G)$.

LEMMA 4. *A group G is an I-group if and only if, for any $g \in G$, $E(G)$ is generated by ι (the identity of $\mathfrak{E}(G)$), $I(G)$ and $N(g)$.*

PROOF. Let G be an I-group. Let us consider the set of all elements $ng + g\eta$ where g is a fixed element of G, n is running over the rational integers and η runs over $I(G)$. This set is obviously an $\mathfrak{J}(G)$-subgroup which is, because of hypothesis, fully invariant. Therefore each $\varphi \in \mathfrak{E}(G)$ carries this set into itself. Hence for a $\varphi \in \mathfrak{E}(G)$ we may write $g\varphi = ng + g\eta$ for some natural integer n and some $\eta \in I(G)$. Therefore $\varphi - n\iota - \eta \in N(g)$, and $\varphi \in \{\iota, I(G), N(g)\}$, i.e. $E(G)$ is generated by ι, $I(G)$ and $N(g)$. Conversely, let us suppose that $E(G) = \{\iota, I(G), N(g)\}$. Let H be an $I(G)$-subgroup, $g \in H$ and $\varphi \in E(G)$. By hypothesis $\varphi = n\,\iota + \eta_1 + \eta_2$ with suitable $\eta_1 \in I(G)$ and $\eta_2 \in N(g)$. Thus $g\varphi = ng + g\eta_1 + g\eta_2 = ng + g\eta_1 \in H$, which establishes the full invariance of H.

Evidently, G is an I-group if, for all fully invariant subgroups T of G, the equality $T^* = T$ holds (Theorem 2).

LEMMA 5. *For a group G, the following assertions are equivalent:*

(i) *each fully invariant subgroup T of G satisfies $T^* = T$,*

(ii) *to each $g \in G$ there exists an $\eta \in \mathfrak{J}(G)$ such that $g\eta = g$.*

PROOF. First we derive (ii) from (i). By Lemma 1, $g\,\mathfrak{I}(G)$ is fully invariant in G. On account of (i), we have $g \in (gI(G))^* = gI(G)$. Then there exists an $\eta \in I(G)$ such that $g = g\eta$.

In order to prove (i) from (ii), let $g \in T^*$, and let $\eta \in \mathfrak{I}(G)$ be chosen so that $g\eta = g$. Then $g = g\eta \in gI(G) \subseteqq T$, which proves (i).

Now we shall consider some special cases.

COROLLARY 1. *If, for some G, $I(G)$ coincides with $E(G)$, then a subgroup of G is an ideal in all rings over G if and only if it is fully invariant in G.*

PROOF. If $I(G) = E(G)$, then Lemma 5 (ii) is satisfied, hence (i) and Theorem 2 imply the statement.

In particular, the hypothesis of Corollary 1 is satisfied if G has a homomorphism onto the infinite cyclic group.

LEMMA 6. *If G is a torsion group, then $I(G)$ is the (maximal) torsion subgroup of $E(G)$.*

PROOF. All homomorphic images of a torsion group are again torsion groups, therefore $I(G)$ is contained in the torsion subgroup of $E(G)$. Now, let η be an element of $E(G)$ of order n. Thus $n(G\eta) = G(n\eta) = 0$, i.e. $G\eta$ is bounded, and in view of the theorem of Prüfer-Baer it is a direct sum of cyclic groups. Let k be the least common multiple of the orders of the cyclic direct summands of $G\eta$. Because of $n(G\eta) = 0$, we must have $k|n$, and because of $G(k\eta) = 0$, $k\eta = 0$ we get $n|k$, that is, $k = n$. Therefore $G\eta$ has a cyclic direct summand G' of order n. Let π be the natural homomorphism of G onto G', and σ the homomorphism $G \to E(G)$ mapping a generator of G' onto η. Then $\eta\pi\sigma$ maps some element of G onto η, i.e. $\eta \in I(G)$. Consequently, $I(G)$ is the (maximal) torsion subgroup of $E(G)$.

COROLLARY 2. *For a torsion group G, the following assertions are equivalent:*

(i) *each fully invariant subgroup T of G satisfies $T^* = T$;*

(ii) $0^* = 0$.

PROOF. (i) evidently implies (ii).

Let $0^* = 0$. Because of Remark 3 after Theorem 2, $G^1 = 0^* = 0$. This implies that any element of G is contained in a finite direct summand of G. Thus the order of the projection onto this direct summand is finite. By Lemma 6, this projection is an element of $I(G)$. Lemma 5 (ii) completes the proof.

It is straightforward to prove:

COROLLARY 3. *If G is torsion-free and divisible, then a subgroup of G is an ideal of all rings over G if and only if it is fully invariant.*

Let us mention the following questions:

PROBLEM 1. Which groups G (besides torsion groups) have the property that $I(G)$ is not only generated by the homomorphic images of G into $E(G)$, but it is exactly their set union?

PROBLEM 2. Is a torsion group G an I-group if and only if $G^1(= 0^*)$ is isomorphic with a subgroup of all roots of unity?

Finally, I wish to thank Prof. L. Fuchs and the members of his seminary for their remarks and advice.

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SOME GENERALIZATIONS OF THE EXACT SEQUENCES CONCERNING HOM AND EXT

By

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We shall consider throughout Abelian groups; these will be called simply groups and denoted by capitals $A, B, \ldots$ The group $\mathrm{Hom}(A, B)$ of all homomorphisms of A into B and the group $\mathrm{Ext}(A, B)$ of all extensions of B by A have the well-known properties[1] that if

$$0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0 \tag{1}$$

is an exact sequence of groups A, B, C and homomorphisms $\alpha, \beta,$ then for any group X, there exist exact sequences

$$\begin{aligned} 0 \to \mathrm{Hom}\,(X, A) \xrightarrow{\alpha^*} \mathrm{Hom}\,(X, B) \xrightarrow{\beta^*} \mathrm{Hom}\,(X, C) \xrightarrow{\gamma} \\ \to \mathrm{Ext}\,(X, A) \xrightarrow{\alpha^*} \mathrm{Ext}\,(X, B) \xrightarrow{\beta^*} \mathrm{Ext}\,(X, C) \to 0 \end{aligned} \tag{2}$$

and

$$\begin{aligned} 0 \to \mathrm{Hom}\,(C, X) \xrightarrow{\beta^*} \mathrm{Hom}\,(B, X) \xrightarrow{\alpha^*} \mathrm{Hom}\,(A, X) \xrightarrow{\delta} \\ \to \mathrm{Ext}\,(C, X) \xrightarrow{\beta^*} \mathrm{Ext}\,(B, X) \xrightarrow{\alpha^*} \mathrm{Ext}\,(A, X) \to 0 \end{aligned} \tag{3}$$

where α^* and β^* are homomorphisms induced by α and β, and γ, δ are homomorphisms depending on (1) [the latter are called connecting homomorphisms].

There are several generalizations of Hom and Ext and exact sequences (2) and (3). Harrison [6] introduced the group of pure extensions, Pext (A, B), and proved that if (1) is pure exact, then (2) and (3) continue to hold even if Ext is replaced by Pext throughout. Pierce [8] has defined the notion of small homomorphisms of p-groups, and has shown that they form a group $\mathrm{Hom}_s(A, B)$ for which the first half of (2) and (in case

[1] For the definitions and basic facts concerning Hom and Ext we refer to Cartan [1], for those on Abelian groups to Fuchs [3].

(1) is pure exact) the first half of (3) hold. Pogány [9] considered those homomorphisms of p-groups whose images are countable; he established exact sequences for these like the first halves of (2) and (3). The group Shom(A, B) of all homomorphisms of A into B whose kernels are essential subgroups of A has been introduced by Harrison, Irwin, Peercy and Walker [7] when they discussed the group Hext of high extensions. For neat exact (1) the first halves of (2) and (3) can be established for this Shom.[2] Our intention is to show that all these results (and some more) may be derived from a common source.

The basic idea is to associate with each Abelian group A an ideal $\mathbf{I}_A$ or a dual ideal $\mathbf{D}_A$ of the lattice $\mathbf{L}(A)$ of its subgroups, and to consider the pairs $(A, \mathbf{I}_A)$ or $(A, \mathbf{D}_A)$ as our objects. In this way the categories $(\mathcal{A}, \mathbf{I})$ and $(\mathcal{A}, \mathbf{D})$ arise where $\mathcal{A}$ denotes the category of all Abelian groups. Then $\mathrm{Hom}_{\mathbf{I}}(A, B)$ and $\mathrm{Hom}_{\mathbf{D}}(A, B)$ will be defined as the sets of all morphisms of these categories from A into B. It turns out that these are subgroups of Hom (A, B), which give rise to left exact functors of $(\mathcal{A}, \mathbf{I})$ and $(\mathcal{A}, \mathbf{D})$ into $\mathcal{A}$. The derived functors will be determined only if the first variable is kept fixed. It can be described as follows: $\mathrm{Ext}_{\mathbf{I}}(A, B)$ is a factor group of the group Fact(A, B) of all factor sets of A into B and has Ext(A, B) for its homomorphic image; and a similar situation holds for $\mathbf{D}$.

In terms of $(\mathcal{A}, \mathbf{I})$ and $(\mathcal{A}, \mathbf{D})$ certain subgroups of Hom can be defined: $\mathrm{Hom}(A|\mathbf{D}, B|\mathbf{I})$ is the set of all homomorphisms η of A into B such that $\mathrm{Ker}\ \eta \in \mathbf{D}_A$ and $\mathrm{Im}\ \eta \in \mathbf{I}_B$. For fixed systems $\mathbf{I}$, $\mathbf{D}$, they induce left exact functors of $\mathcal{A}$ into $\mathcal{A}$, and the derived functors can be characterized in terms of factor sets. In this case, moreover, the derived functors can be represented as direct limits of systems obtained from Ext.

Finally, we introduce several subgroups of Ext. $\mathrm{Ext}(A|\mathbf{I}, B)$ and $\mathrm{Ext}(A|\mathbf{D}, B)$ will be defined as the sets of all (classes of) extensions of B by A, which split whenever we restrict them to arbitrary $A^* \in \mathbf{I}_A$ and to suitably chosen $A^* \in \mathbf{D}_A$, respectively. $\mathrm{Ext}(A, B|\mathbf{I})$ and $\mathrm{Ext}(A, B|\mathbf{D})$ are defined analogously by the requirement that the extensions split on passing modulo some $B^* \in \mathbf{I}_B$ and modulo any $B^* \in \mathbf{D}_B$, respectively. Our results will show that if we start with an $\mathbf{I}$-exact or $\mathbf{D}$-exact sequence (1) and assume that (1) is an extension of A by C of special kind, then exact sequences like (2) and (3) can be established with Ext replaced by $\mathrm{Ext}(\ |\mathbf{I},\ )$ and by $\mathrm{Ext}(\ ,\ |\mathbf{D})$, respectively.

Some problems which arose in connection with our discussions will be dealt with in a subsequent publication.

Let us note that some of our results can be generalized to unitary modules over rings with identity.

[2] For other generalizations of (2), (3) see Fuchs [4].

§ 1. The categories $(\mathscr{A}, \mathbf{I})$ and $(\mathscr{A}, \mathbf{D})$

Let $\mathscr{A}$ denote the category of Abelian groups. We select for each $A \in \mathscr{A}$ an ideal $\mathbf{I}_A$ of the lattice $\mathbf{L}(A)$ of all subgroups of A. We define the *category* $(\mathscr{A}, \mathbf{I})$ to consist of the *objects* $(A, \mathbf{I}_A)$ for $A \in \mathscr{A}$. The *morphisms*

$$\varphi : (A, \mathbf{I}_A) \to (B, \mathbf{I}_B)$$

of this category are the homomorphisms[3] of A into B which map subgroups in $\mathbf{I}_A$ into subgroups in $\mathbf{I}_B$:

$$A\,\varphi \subseteq B \quad \text{and} \quad \mathbf{I}_A\,\varphi \subseteq \mathbf{I}_B.$$

Evidently, the product of two morphisms is again one. By a *monomorphism* of this category we mean a morphism $\varphi : (A, \mathbf{I}_A) \to (B, \mathbf{I}_B)$ such that φ acts as a monomorphism of A into B and

$$\{A'\,\varphi \mid A' \in \mathbf{I}_A\} = \{B' \cap A\,\varphi \mid B' \in \mathbf{I}_B\}.$$

An *epimorphism* of $(A, \mathbf{I})$ is a morphism φ, which acts as an epimorphism of A onto B and

$$\mathbf{I}_A\,\varphi = \mathbf{I}_B.$$

Given the category $(\mathscr{A}, \mathbf{I})$, we say that the sequence

$$0 \to (A, \mathbf{I}_A) \xrightarrow{\alpha} (B, \mathbf{I}_B) \xrightarrow{\beta} (C, \mathbf{I}_C) \to 0$$

is *exact*, or simply that the sequence

$$0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0 \tag{4}$$

is $\mathbf{I}$-*exact* if it is exact and if α is a monomorphism and β is an epimorphism in the category $(\mathscr{A}, \mathbf{I})$.

Examples.—1. The category $(\mathscr{A}, \mathbf{L})$ where the ideal $\mathbf{L}_A$ is equal to $\mathbf{L}(A)$.—Now the morphisms are arbitrary homomorphisms between Abelian groups; monomorphisms and epimorphisms have the usual meaning.

2. The category $(\mathscr{A}, \mathbf{F})$ where the ideal $\mathbf{F}_A$ is the set of all finitely generated subgroups of A.—Again, morphisms are arbitrary homomorphisms and monomorphisms, epimorphisms are simply monomorphisms and epimorphims in the usual sense.

3. The category $(\mathscr{A}, \mathbf{B})$ where $\mathbf{B}_A$ denotes the ideal of all bounded subgroups of A.—Morphisms are now simply homomorphisms, monomorphisms have again the usual meaning, but epimorphisms are not the usual epimorphisms.

[3] The mappings will be written from the right and thus the product $\alpha\beta$ of two mappings α, β is obtained by first performing α and then β.

4. Let $(\mathscr{A}, \mathbf{T})$ be the category where the ideal $\mathbf{T}_A$ consists of all torsion subgroups of A.—For the maps the same holds as in the previous example.

5. The category $(\mathscr{A}, \mathbf{I}^{(\mathfrak{m})})$ where $\mathbf{I}_A^{(\mathfrak{m})}$ consists of all subgroups of cardinality $< \mathfrak{m}$ of A.—Here morphisms are the same as homomorphisms, and monomorphisms, epimorphisms have the usual meaning.

6. Let $(\mathscr{A}, \mathbf{U})$ denote the category where the objects are all pairs $(A, \mathbf{I}_A)$ with $A \in \mathscr{A}$ and $\mathbf{I}_A$ running over all ideals of $\mathbf{L}(A)$.

Next let us choose a dual ideal $\mathbf{D}_A$ in the lattice $\mathbf{L}(A)$ for each $A \in \mathscr{A}$, and define the *category* $(\mathscr{A}, \mathbf{D})$ to consist of the *objects* $(A, \mathbf{D}_A)$ for all $A \in \mathscr{A}$ and of the *morphisms*

$$\psi : (A, \mathbf{D}_A) \to (B, \mathbf{D}_B)$$

where ψ: $A \to B$ is a group homomorphism such that

to each $B^* \in \mathbf{D}_B$ there exists an $A^* \in \mathbf{D}_A$ satisfying $A^* \psi \subseteq B^*$.

The morphism ψ is a *monomorphism* if ψ: $A \to B$ is a monomorphism and

$$\{A' \psi \mid A' \in \mathbf{D}_A\} = \{B' \cap A \psi \mid B' \in \mathbf{D}_B\};$$

and it is an *epimorphism* if ψ: $A \to B$ is an epimorphism and

$$\mathbf{D}_A \psi = \mathbf{D}_B .$$

The definition of the *exact*ness of

$$0 \to (A, \mathbf{D}_A) \to (B, \mathbf{D}_B) \to (C, \mathbf{D}_C) \to 0$$

and the $\mathbf{D}$-*exact*ness of $0 \to A \to B \to C \to 0$ is evident.

Examples.—7. The category $(\mathscr{A}, \mathbf{D})$ is essentially the same as the one given in Example 1 if $\mathbf{D}_A = \mathbf{L}(A)$ for all A.

8. Let the category $(\mathscr{A}, \mathbf{G})$ be defined by putting $\mathbf{G}_A$ equal to the set of all subgroups A' of A, which are of finite index in A (or the factor group A/A' is finitely generated).—Now morphisms are group homomorphisms and epimorphisms have their usual meaning.

9. The category $(\mathscr{A}, \mathbf{C})$ where $\mathbf{C}_A$ is the dual ideal of all subgroups A' of A such that A/A' is bounded.—In this case the epimorphisms have the usual meaning. A monomorphism of this category is a monomorphism of A onto a subgroup of B whose own n-adic topology coincides with the topology induced by the n-adic topology of B.

10. The category $(\mathscr{A}, \mathbf{W})$ where $\mathbf{W}_A$ is the dual ideal of all subgroups A' of A for which A/A' is a torsion group. —The monomorphisms and epimorphisms have the same meaning as in $\mathscr{A}$.

11. The category $(A, \mathbf{D}^{(\mathfrak{m})})$ where $\mathbf{D}_A^{(\mathfrak{m})}$ denotes the dual ideal of all subgroups of index $< \mathfrak{m}$ of A.

12. Let $(\mathscr{A}, \mathbf{V})$ denote the category whose objects are all pairs $(A, \mathbf{D}_A)$ where $A \in \mathscr{A}$ and $\mathbf{D}_A$ ranges over all dual ideals of $\mathbf{L}(A)$.

13. Let $(\mathscr{A}, \mathbf{S})$ be the category consisting of all objects $(T, \mathbf{S}_T)$ where T is a torsion group and $\mathbf{S}_T$ is the dual ideal of all subgroups that contain large subgroups in the sense of Pierce [8].

14. Let $(\mathscr{A}, \mathbf{H})$ mean the category with $\mathbf{H}_A$ the set of all essential subgroups of A. Morphisms are nothing else than homomorphisms; and the monomorphisms, but not the epimorphisms have their usual meaning.

Evidently, there are natural functors $(\mathscr{A}, \mathbf{I}) \to \mathscr{A}$ and $(\mathscr{A}, \mathbf{D}) \to \mathscr{A}$ given by simple omitting $\mathbf{I}_A$ and $\mathbf{D}_A$ throughout.

There is no difficulty in defining $\mathrm{Hom}_{\mathbf{I}}(A, B)$ as the set of all morphisms $(A, \mathbf{I}_A) \to (B, \mathbf{I}_B)$ in the category $(\mathscr{A}, \mathbf{I})$ with addition defined as in the case of $\mathscr{A}$. Evidently, $\mathrm{Hom}_{\mathbf{I}}(A, B)$ is always an Abelian group and is, in the natural way, isomorphic to a subgroup of $\mathrm{Hom}(A, B)$. In Examples 1—4, this subgroup is not a proper one.

Similarly, $\mathrm{Hom}_{\mathbf{D}}(A, B)$ can be defined as the set of all morphisms $(A, \mathbf{D}_A) \to (B, \mathbf{D}_B)$ in $(\mathscr{A}, \mathbf{D})$ with addition as in $\mathscr{A}$. It may also be regarded as a subset of $\mathrm{Hom}(A, B)$. It is a subgroup, for the zero homomorphism belongs to it, and if $\eta_1, \eta_2 \in \mathrm{Hom}_{\mathbf{D}}(A, B)$, and if given a $B^* \in \mathbf{D}_B$, then there exist $A_1^*, A_2^* \in \mathbf{D}_A$ such that $A_1^* \eta_1, A_2^* \eta_2 \subseteq B^*$ whence $A^* = A_1^* \cap \cap A_2^* \in \mathbf{D}_A$ satisfies $A^*(\eta_1 - \eta_2) \subseteq B^*$, and so $\eta_1 - \eta_2 \in \mathrm{Hom}_{\mathbf{D}}(A, B)$.

The morphism $\alpha\colon (A, \mathbf{I}_A) \to (B, \mathbf{I}_B)$ induces maps

$$\mathrm{Hom}_{\mathbf{I}}(X, A) \to \mathrm{Hom}_{\mathbf{I}}(X, B) \text{ and } \mathrm{Hom}_{\mathbf{I}}(B, X) \to \mathrm{Hom}_{\mathbf{I}}(A, X),$$

for an arbitrary group X, given by $\eta \to \eta\alpha$ and $\chi \to \alpha\chi$, respectively. It is readily checked that they are in fact homomorphisms, and so are the maps defined analogously for the category $(\mathscr{A}, \mathbf{D})$.

For these groups $\mathrm{Hom}_{\mathbf{I}}$ and $\mathrm{Hom}_{\mathbf{D}}$ it is easy to verify the analogues of the classical exact sequences:

THEOREM 1. *If* $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ *is an* $\mathbf{I}$*-exact sequence, then for an arbitrary group* X *we have the induced exact sequences*[4]

$$0 \to \mathrm{Hom}_{\mathbf{I}}(X, A) \xrightarrow{\alpha^*} \mathrm{Hom}_{\mathbf{I}}(X, B) \xrightarrow{\beta^*} \mathrm{Hom}_{\mathbf{I}}(X, C)$$

and

$$0 \to \mathrm{Hom}_{\mathbf{I}}(C, X) \xrightarrow{\beta^*} \mathrm{Hom}_{\mathbf{I}}(B, X) \xrightarrow{\alpha^*} \mathrm{Hom}_{\mathbf{I}}(A, X).$$

We know that $\mathrm{Hom}_{\mathbf{I}}(X, A)$ is a subgroup of $\mathrm{Hom}(X, A)$ and $\mathrm{Hom}_{\mathbf{I}}(X, B)$ is a subgroup of $\mathrm{Hom}(X, B)$, furthermore that the homomorphism induced by α between $\mathrm{Hom}(X, A)$ and $\mathrm{Hom}(X, B)$ is a monomorphism. Since our α^* is the restriction of this to $\mathrm{Hom}_{\mathbf{I}}(X, A)$, it is also a monomorphism. We

[4] No ambiguity will arise if we denote throughout by φ^* the mapping induced by φ.

need to show that Im $\alpha^* = \text{Ker } \beta^*$. Since $\alpha\beta = 0$, the definition of induced homomorphism implies $\alpha^*\beta^* = 0$, and therefore it suffices to show that Ker $\beta^* \subseteq$ Im α^*. Let $\eta \in$ Ker β^*, then $\eta\beta = 0$, and the exact sequence (2) guarantees the existence of a $\chi \in \text{Hom}(X, A)$ such that $\eta = \chi\alpha$. For any $X^* \in \mathbf{I}_X$, the image $X^*\eta = B^*$ belongs to $\mathbf{I}_B$ and is contained in $A\alpha$, whence the **I**-exactness of the original sequence implies that B^* is of the form $B^* = A^*\alpha$ for some $A^* \in \mathbf{I}_A$. Now $X^*\chi\alpha = A^*\alpha$ implies $X^*\chi = A^*$, α being a monomorphism, i.e. $\chi \in \text{Hom}_{\mathbf{I}}(X, A)$.

In order to prove the exactness of the second sequence, we conclude by the same argument that now β^* is a monomorphism and that it is sufficient to prove the inclusion Ker $\alpha^* \subseteq$ Im β^*. Let $\eta \in$ Ker α^*; then from the exactness of (3) it follows that there is a $\chi \in \text{Hom}(C, X)$ satisfying $\eta = \beta\chi$. The hypothesis of **I**-exactness implies that given a $C^* \in \mathbf{I}_C$, there is a $B^* \in \mathbf{I}_B$ such that $C^* = B^*\beta$. But then $C^*\chi = B^*\beta\chi = B^*\eta = X^*$ belongs to $\mathbf{I}_X$, and thus χ belongs to $\text{Hom}_{\mathbf{I}}(C, X)$, which completes the proof.

THEOREM 2. *If* $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ *is a* **D**-*exact sequence, then the induced sequences*

$$0 \to \text{Hom}_{\mathbf{D}}(X, A) \xrightarrow{\alpha^*} \text{Hom}_{\mathbf{D}}(X, B) \xrightarrow{\beta^*} \text{Hom}_{\mathbf{D}}(X, C)$$

and

$$0 \to \text{Hom}_{\mathbf{D}}(C, X) \xrightarrow{\beta^*} \text{Hom}_{\mathbf{D}}(B, X) \xrightarrow{\alpha^*} \text{Hom}_{\mathbf{D}}(A, X)$$

are exact for every group X.

The proof is similar to that of Theorem 1; we leave it to the reader.

Now we have come to the problem of determining the derived functors of $\text{Hom}_{\mathbf{I}}$ and $\text{Hom}_{\mathbf{D}}$. If we wish to mean by an **I**-extension (**D**-extension) of A by C an **I**-exact (**D**-exact) sequence (4), then the set of all **I**-extensions (**D**-extensions) may be empty, since the categories $(\mathcal{A}, \mathbf{I})$ and $(\mathcal{A}, \mathbf{D})$ need not be additive. Therefore the derived functors cannot be defined in terms of **I**- or **D**-extensions. But they can be defined by making use of factor sets just as in the case of Hom; the only difference is that the equivalence of factor sets must be taken in a weaker sense.

Let A, B be two groups in the category $(\mathcal{A}, \mathbf{I})$ or $(\mathcal{A}, \mathbf{D})$. A function

$$\xi : a \to \xi(a) \in B \qquad (a \in A)$$

of A into B will be said to be an **I**- or **D**-*function* if it maps subgroups $A^* \in \mathbf{I}_A$ into subgroups $B^* \in \mathbf{I}_B$, and if to any $B^* \in \mathbf{D}_B$ there is some $A^* \in \mathbf{D}_A$ so that ξ maps A^* into B^*, respectively. (Thus a morphism is a homomorphism which is at the same time an **I**- or **D**-function.) It is easy to check that, for fixed A and B, the **I**-functions of A into B form an Abelian group under the obvious addition; the same holds of course for the **D**-functions too.

By an **I**- or **D**-*transformation set* of A into B we understand a transformation set

$$f(a_1, a_2) = \xi(a_1) + \xi(a_2) - \xi(a_1 + a_2)$$

where ξ is an **I**- or **D**-function. These form a group under addition which will be denoted by $\mathrm{Trans}_{\mathbf{I}}(A, B)$ and $\mathrm{Trans}_{\mathbf{D}}(A, B)$, respectively. They are obviously subgroups of $\mathrm{Trans}(A, B)$, the group of all transformation sets of A into B.

Now $\mathrm{Ext}_{\mathbf{I}}(A, B)$ will be defined as the factor group of the group $\mathrm{Fact}\,(A, B)$ of all factor sets of A into B modulo the subgroup $\mathrm{Trans}_{\mathbf{I}}(A, B)$:

$$\mathrm{Ext}_{\mathbf{I}}\,(A, B) = \mathrm{Fact}\,(A, B)/\mathrm{Trans}_{\mathbf{I}}\,(A, B).$$

A similar definition applies to $\mathrm{Ext}_{\mathbf{D}}(A, B)$:

$$\mathrm{Ext}_{\mathbf{D}}\,(A, B) = \mathrm{Fact}\,(A, B)/\mathrm{Trans}_{\mathbf{D}}\,(A, B).$$

We shall restrict ourselves to considering $\mathrm{Ext}_{\mathbf{I}}$ only, since $\mathrm{Ext}_{\mathbf{D}}$ can be treated similarly.

A morphism $\alpha\colon (A, \mathbf{I}_A) \to (B, \mathbf{I}_B)$ induces, for any group X, homomorphisms

$$\mathrm{Ext}_{\mathbf{I}}\,(X, A) \to \mathrm{Ext}_{\mathbf{I}}\,(X, B) \quad \text{and} \quad \mathrm{Ext}_{\mathbf{I}}\,(B, X) \to \mathrm{Ext}_{\mathbf{I}}\,(A, X)$$

defined by $f(x_1, x_2) \to f(x_1, x_2)\alpha$ and $f(b_1, b_2) \to f(a_1\,\alpha, a_2\,\alpha)$, respectively.[5] A straightforward calculation shows that these are independent of the choice of f mod $\mathrm{Trans}_{\mathbf{I}}$ and are additive, so that they are in fact homomorphisms between the indicated groups.

We still need to mention the connecting homomorphisms. Let the extension B of A by C in (4) be represented by the factor set $g(c_1, c_2)$ (with values in A; $c_i \in C$). Recall that g is obtained from a function $c \to \xi(c)$ of C into B where $\xi\,(c)$ is an element of the coset c (mod A):

$$g(c_1, c_2) = \xi(c_1) + \xi(c_2) - \xi(c_1 + c_2). \tag{5}$$

Observe that if (4) is **I**-exact then ξ may be chosen as an **I**-function of C into B.[6] By making use of this g the connecting homomorphisms

$$\gamma\colon \mathrm{Hom}_{\mathbf{I}}(X, C) \to \mathrm{Ext}_{\mathbf{I}}(X, A), \quad \delta\colon \mathrm{Hom}_{\mathbf{I}}(A, X) \to \mathrm{Ext}_{\mathbf{I}}(C, X)$$

are defined by

$$\gamma : \eta \to g(x_1\,\eta, x_2\,\eta), \qquad \delta : \chi \to g(c_1, c_2)\,\chi. \tag{6}$$

These are actually homomorphisms, since the **I**-exactness of (4) ensures that for another factor set $g'(c_1, c_2) = \xi'(c_1) + \xi'(c_2) - \xi'(c_1 + c_2)$ (with

[5] We denote factor sets by f, g, h depending on two arguments.

[6] This can be shown by making use of Zorn's lemma. We shall always think of g as obtained from an **I**-function. For **D**-functions the assertion is obvious.

an **I**-function ξ') representing (4) the difference $g - g'$ will belong to $\mathrm{Trans}_{\mathbf{I}}(C, A)$, because $c \to \xi(c) - \xi'(c)$ is a function of C into $A\alpha$ mapping a subgroup $C^* \in \mathbf{I}_C$ into some $B^* \in \mathbf{I}_B$ and hence into $B^* \cap A\alpha = A^*\alpha$ with $A^* \in \mathbf{I}_A$. From this it is readily concluded that γ and δ are homomorphisms between the indicated groups.

After these we are ready to state and prove the main results of this section.

THEOREM 3. *If* (4) *is an* **I**-*exact sequence represented by the factor set* (5), *then, for any group* X, *with the induced homomorphisms and connecting homomorphism* (6) *the sequence*

$$(7) \qquad 0 \to \mathrm{Hom}_{\mathbf{I}}(X, A) \xrightarrow{\alpha^*} \mathrm{Hom}_{\mathbf{I}}(X, B) \xrightarrow{\beta^*} \mathrm{Hom}_{\mathbf{I}}(X, C) \xrightarrow{\gamma}$$

$$\to \mathrm{Ext}_{\mathbf{I}}(X, A) \xrightarrow{\alpha^*} \mathrm{Ext}_{\mathbf{I}}(X, B) \xrightarrow{\beta^*} \mathrm{Ext}_{\mathbf{I}}(X, C) \to 0$$

is exact.

In view of Theorem 1, the exactness has to be established only at the last parts of (7). It is routine to check that the composite maps are zero throughout.

Let $\eta \in \mathrm{Ker}\,\gamma$. This means that $g(x_1\eta, x_2\eta)$ is an **I**-transformation set, $g(x_1\eta, x_2\eta) = a(x_1) + a(x_2) - a(x_1 + x_2)$ for some **I**-function $x \to a(x)$ of X into $A\alpha$. By virtue of (5), $\xi(x_1\eta) + \xi(x_2\eta) - \xi((x_1 + x_2)\eta) = a(x_1) + a(x_2) - a(x_1 + x_2)$ whence $\epsilon\colon x \to \xi(x\eta) - a(x)$ is a homomorphism of X into B. It is an **I**-function (as the difference of two **I**-functions), and thus it belongs to $\mathrm{Hom}_{\mathbf{I}}(X, B)$. From $x\epsilon\beta = \xi(x\eta)\beta - a(x)\beta = x\eta$ we see that $\epsilon\beta = \eta$, and so (7) is exact at $\mathrm{Hom}_{\mathbf{I}}(X, C)$.

Let $f(x_1, x_2)$ belong to $\mathrm{Ker}\,\alpha^*$, i.e. $f(x_1, x_2)\alpha = b(x_1) + b(x_2) - b(x_1 + x_2)$ for some **I**-function $x \to b(x)$ of X into B. Define $\eta\colon x \to b(x)\beta$ which is obviously an **I**-function, and in view of $f(x_1, x_2)\alpha \in A\alpha$ it is a homomorphism whence $\eta \in \mathrm{Hom}_{\mathbf{I}}(X, C)$. Thus $x \to b(x) - \xi(x\eta)$ is an **I**-function of X into B, consequently, $f(x_1, x_2)\alpha - g(x_1\eta, x_2\eta)$ is an **I**-transformation set and γ maps η upon f.

Next let $f(x_1, x_2) \in \mathrm{Ker}\,\beta^*$, that is, $f(x_1, x_2)\beta = c(x_1) + c(x_2) - c(x_1 + x_2)$ for some **I**-function $x \to c(x)$ of X into C. Then $f(x_1, x_2) - \xi(c(x_1)) - \xi(c(x_2)) + \xi(c(x_1 + x_2))$ has values in $A\alpha$, and thus the application of α^{-1} yields a factor set $h(x_1, x_2)$ of X into A mapped by α into $f(x_1, x_2)$ mod $\mathrm{Trans}_{\mathbf{I}}(X, B)$. Hence $\mathrm{Ker}\,\beta^* \subseteq \mathrm{Im}\,\alpha^*$.

If $f(x_1, x_2)$ denotes now an arbitrary element of $\mathrm{Ext}_{\mathbf{I}}(X, C)$, then, because of (2), there exists a factor set $h(x_1, x_2)$ satisfying $h(x_1, x_2)\beta = f(x_1, x_2) + c(x_1) + c(x_2) - c(x_1 + x_2)$ for some function $x \to c(x)$ of X into C. If $b(x) \in B$ satisfies $b(x)\beta = c(x)$, then the factor set $h(x_1, x_2) - b(x_1) - b(x_2) + b(x_1 + x_2)$ is mapped by β^* upon $f(x_1, x_2)$ whence β^* is an epimorphism.

A similar method of proof applies to establish:

THEOREM 4. *If* (4) *is a* **D**-*exact sequence, then for any group* X, *the following induced sequence is exact:*

$$(8)\qquad 0 \to \mathrm{Hom}_{\mathbf{D}}(X, A) \to \mathrm{Hom}_{\mathbf{D}}(X, B) \to \mathrm{Hom}_{\mathbf{D}}(X, C) \to$$
$$\to \mathrm{Ext}_{\mathbf{D}}(X, A) \to \mathrm{Ext}_{\mathbf{D}}(X, B) \to \mathrm{Ext}_{\mathbf{D}}(X, C) \to 0.$$

The problem naturally arises: how can (3) be carried over to this case? The author has been unable to answer this question satisfactorily. The reason of this seems to lie in the fact that, without any additional assumption on $(\mathcal{A}, \mathbf{I})$ and $(\mathcal{A}, \mathbf{D})$, **I**- and **D**-exact sequences cannot be described by making use of factor sets only. In fact, if we assume additional conditions on the behaviour of **I**- and **D**-exact sequences, then a suitable definition of **I**- and **D**-factor sets leads to an exact sequence of the form (3). This will not be discussed here.

§ 2. Certain subgroups of Hom

With the aid of two of the categories $\mathcal{A}$, $(\mathcal{A}, \mathbf{I})$ and $(\mathcal{A}, \mathbf{D})$ we shall define three types of homomorphism groups which are all subgroups of the homomorphism groups in the classical sense.

Let $A \in \mathcal{A}$ and let $(B, \mathbf{I}_B)$ belong to the category $(\mathcal{A}, \mathbf{I})$. Consider the set of all homomorphisms $\eta : A \to B$ (in the category $\mathcal{A}$) for which

$$(9)\qquad \mathrm{Im}\,\eta \in \mathbf{I}_B\,.$$

This set will be denoted by $\mathrm{Hom}(A, B|\mathbf{I})$.

Next let $(A, \mathbf{D}_A) \in (\mathcal{A}, \mathbf{D})$ and $B \in \mathcal{A}$. The set of all homomorphisms η: $A \to B$ such that

$$(10)\qquad \mathrm{Ker}\,\eta \in \mathbf{D}_A$$

will be denoted by $\mathrm{Hom}(A|\mathbf{D}, B)$.

Finally, $\mathrm{Hom}(A|\mathbf{D}, B|\mathbf{I})$ will denote the set of all homomorphisms η: $A \to B$ for which both inclusions hold.

These definitions imply that if[7] $\mathbf{I} \subseteq \mathbf{J}$ and $\mathbf{D} \subseteq \mathbf{E}$ then $\mathrm{Hom}(A|\mathbf{D}, B|\mathbf{I}) \subseteq \mathrm{Hom}(A|\mathbf{E}, B|\mathbf{J})$. We have the elementary but important theorem:

THEOREM 5. *The sets* $\mathrm{Hom}(A, B|\mathbf{I})$, $\mathrm{Hom}(A|\mathbf{D}, B)$ *and* $\mathrm{Hom}(A|\mathbf{D}, B|\mathbf{I})$ *are subgroups of* $\mathrm{Hom}(A, B)$ *satisfying*

$$\mathrm{Hom}(A, B|\mathbf{I}) \cap \mathrm{Hom}(A|\mathbf{D}, B) = \mathrm{Hom}(A|\mathbf{D}, B|\mathbf{I})\,.$$

[7] By $\mathbf{I} \subseteq \mathbf{J}$ we mean that $\mathbf{I}_A \subseteq \mathbf{J}_A$ for all $A \in \mathcal{A}$. Here and in the sequel **I** and **J** denote systems of ideals, **D** and **E** denote systems of dual ideals.

The validity of the formula is obvious in view of the definitions. The subgroup properties of $\mathrm{Hom}(A, B|\mathbf{I})$ and $\mathrm{Hom}(A|\mathbf{D}, B)$ follow from the fact that the zero homomorphism belongs to them and if η_1, η_2 are two homomorphisms of A into B, then

$$\mathrm{Im}\,(\eta_1 - \eta_2) \subseteq \mathrm{Im}\,\eta_1 \cup \mathrm{Im}\,\eta_2\,, \qquad \mathrm{Ker}\,(\eta_1 - \eta_2) \supseteq \mathrm{Ker}\,\eta_1 \cap \mathrm{Ker}\,\eta_2$$

hold.

Next let $\varphi\colon (A^*, \mathbf{D}_{A^*}) \to (A, \mathbf{D}_A)$ and $\psi\colon (B, \mathbf{I}_B) \to (B^*, \mathbf{I}_{B^*})$ be morphisms of the respective categories. They induce a homomorphism

$$\mathrm{Hom}\,(\varphi, \psi): \ \mathrm{Hom}\,(A|\mathbf{D}, B|\mathbf{I}) \to \mathrm{Hom}\,(A^*|\mathbf{D}, B^*|\mathbf{I})$$

defined in the usual way as $\eta \to \varphi\eta\psi$. Here $\mathrm{Im}\,\varphi\eta\psi$ is contained in $\mathrm{Im}\,\eta\psi$ which lies in $\mathbf{I}_B\,\psi \subseteq \mathbf{I}_{B^*}$, and if $A_0^* \in \mathbf{D}_{A^*}$ satisfies $A_0^*\,\varphi \subseteq \mathrm{Ker}\,\eta \in \mathbf{D}_A$ then $\mathrm{Ker}\,\varphi\eta\psi$ contains A_0^*; therefore $\varphi\eta\psi$ in fact belongs to $\mathrm{Hom}(A^*|\mathbf{D}, B^*|\mathbf{I})$.

We have the following two exactness theorems.

THEOREM 6. *If*

$$0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0 \tag{11}$$

is an $\mathbf{I}$-*exact sequence, then for any group* X *and category* $(\mathcal{A}, \mathbf{D})$ *the induced sequence*

$$0 \to \mathrm{Hom}\,(X|\mathbf{D}, A|\mathbf{I}) \xrightarrow{\alpha^*} \mathrm{Hom}\,(X|\mathbf{D}, B|\mathbf{I}) \xrightarrow{\beta^*} \mathrm{Hom}\,(X|\mathbf{D}, C|\mathbf{I})$$

is exact.

By Theorem 5, $\mathrm{Hom}(X|\mathbf{D}, A|\mathbf{I})$ is a subgroup of $\mathrm{Hom}(X, A)$ and $\mathrm{Hom}(X|\mathbf{D}, B|\mathbf{I})$ is a subgroup of $\mathrm{Hom}(X, B)$. Since α^* may be viewed as the restriction of the induced homomorphism of $\mathrm{Hom}(X, A)$ into $\mathrm{Hom}(X, B)$, the exactness of (2) implies that α^* is in fact a monomorphism. Thus what we have to verify is the equality $\mathrm{Im}\,\alpha^* = \mathrm{Ker}\,\beta^*$, or more simply, the inclusion $\mathrm{Ker}\,\beta^* \subseteq \mathrm{Im}\,\alpha^*$, the converse inclusion being obvious from the definition and $\alpha\beta = 0$. If $\eta \in \mathrm{Ker}\,\beta^*$, then $\eta\beta = 0$ and $\mathrm{Im}\,\eta \subseteq \mathrm{Ker}\,\beta = \mathrm{Im}\,\alpha$. Thus $\eta\alpha^{-1}$ is meaningful and belongs to $\mathrm{Hom}(X|\mathbf{D}, A|\mathbf{I})$. Clearly, α^* sends $\eta\alpha^{-1}$ into η, whence $\eta \in \mathrm{Im}\,\alpha^*$, and the proof is completed.

THEOREM 7. *If* (11) *is a* $\mathbf{D}$-*exact sequence, then for each group* X *and category* $(\mathcal{A}, \mathbf{I})$ *the induced sequence*

$$0 \to \mathrm{Hom}\,(C|\mathbf{D}, X|\mathbf{I}) \xrightarrow{\beta^*} \mathrm{Hom}\,(B|\mathbf{D}, X|\mathbf{I}) \xrightarrow{\alpha^*} \mathrm{Hom}\,(A|\mathbf{D}, X|\mathbf{I})$$

is exact.

Again by making use of Theorem 5 and the exactness of (3), we conclude that it suffices to verify that $\mathrm{Im}\,\beta^* = \mathrm{Ker}\,\alpha^*$ or that $\mathrm{Ker}\,\alpha^* \subseteq \mathrm{Im}\,\beta^*$. If $\chi \in \mathrm{Ker}\,\alpha^*$, then $\alpha\chi = 0$. This means that χ maps $\mathrm{Im}\,\alpha = \mathrm{Ker}\,\beta$ onto 0, and thus it induces a homomorphism $\eta\colon C \to X$ such that $c\,\eta = b\,\chi$ for all $c \in C$, $c = b\,\beta$ $(b \in B)$. Here obviously $\chi = \beta\eta$ where $\eta \in \mathrm{Hom}(C|\mathbf{D}, X|\mathbf{I})$ because of $\mathrm{Ker}\,\eta = \mathrm{Ker}\,\chi + A\,\alpha \in \mathbf{D}_C$ and $\mathrm{Im}\,\chi = \mathrm{Im}\,\eta$.

The last two theorems show that $\mathrm{Hom}(X|\mathbf{D},\ )$ and $\mathrm{Hom}(\ ,X|\mathbf{I})$ give rise to left exact functors of $(\mathscr{A}, \mathbf{I})$ and $(\mathscr{A}, \mathbf{D})$, respectively, into $\mathscr{A}$. Our next purpose is to determine the derived functors. First we give a direct limit representation for the Homs under consideration, from which the derived functors can easily be obtained.

An ideal $\mathbf{I}_A$ of A determines a direct system[8] of subgroups of A. This consists of all subgroups A_λ of A belonging to $\mathbf{I}_A$, the indices are ordered by putting $\lambda \leqq \mu$ if and only if $A_\lambda \subseteq A_\mu$, and the maps $\pi_\lambda^\mu\colon A_\lambda \to A_\mu$ are the inclusion maps for $\lambda \leqq \mu$; in particular, π_λ^λ is the identity map of A_λ. If $\alpha\colon (A, \mathbf{I}_A) \to (B, \mathbf{I}_B)$ is a morphism of the category $(\mathscr{A}, \mathbf{I})$, then it induces a map $\bar{\alpha}$ between the two arising direct systems. For, to any pair $\lambda \leqq \mu$ there exists a pair $\lambda' \leqq \mu'$ such that the diagram

$$\begin{array}{ccc} A_\lambda & \to & A_\mu \\ \alpha\downarrow & & \downarrow\alpha \\ B_{\lambda'} & \to & B_{\mu'} \end{array} \tag{12}$$

is commutative. We have:

Lemma 1. *The groups* $\mathrm{Hom}(X|\mathbf{D}, A_\lambda)$ $(A_\lambda \in \mathbf{I}_A)$ *and the groups* $\mathrm{Ext}(X, A_\lambda)$ $(A_\lambda \in \mathbf{I}_A)$ *form direct systems where the homomorphisms are the maps* $\pi_\lambda^{\mu*}\colon \mathrm{Hom}(X|\mathbf{D}, A_\lambda) \to \mathrm{Hom}(X|\mathbf{D}, A_\mu)$ *and* $\mathrm{Ext}(X, A_\lambda) \to \mathrm{Ext}(X, A_\mu)$ *induced by* π_λ^μ. *Any map* φ *between the systems* $\{A_\lambda; \pi_\lambda^\mu\}$ *and* $\{B_{\lambda'}; \varrho_{\lambda'}^{\mu'}\}$ *induces maps between the systems* $\{\mathrm{Hom}(X|\mathbf{D}, A_\lambda)\}$ *and* $\{\mathrm{Hom}(X|\mathbf{D}, B_{\lambda'})\}$, $\{\mathrm{Ext}(X, A_\lambda)\}$ *and* $\{\mathrm{Ext}(X, B_{\lambda'})\}$ *in the obvious manner.*

Since for $\lambda \leqq \mu \leqq \nu$ we have $\pi_\lambda^{\mu*}\,\pi_\mu^{\nu*} = (\pi_\lambda^\mu\,\pi_\mu^\nu)^* = \pi_\lambda^{\nu*}$, and since $\pi_\lambda^{\lambda*}$ is the identity, the first assertion is clear. The second assertion follows from the commutativity of the diagrams

$$\begin{array}{ccc} \mathrm{Hom}(X|\mathbf{D}, A_\lambda) & \to & \mathrm{Hom}(X|\mathbf{D}, A_\mu) \\ \downarrow & & \downarrow \\ \mathrm{Hom}(X|\mathbf{D}, B_{\lambda'}) & \to & \mathrm{Hom}(X|\mathbf{D}, B_{\mu'}) \end{array} \qquad \begin{array}{ccc} \mathrm{Ext}(X, A_\lambda) & \to & \mathrm{Ext}(X, A_\mu) \\ \downarrow & & \downarrow \\ \mathrm{Ext}(X, B_{\lambda'}) & \to & \mathrm{Ext}(X, B_{\mu'}) \end{array}$$

with maps induced by those of (12).

A dual ideal $\mathbf{D}_A$ of A defines an inverse system of factor groups of A. This system consists of all A/A_λ with $A_\lambda \in \mathbf{D}_A$ where the index set is ordered so that $\lambda \leqq \mu$ if and only if $A_\lambda \supseteq A_\mu$, and the maps for $\lambda \leqq \mu$ are $\varphi_\mu^\lambda\colon A/A_\mu \to A/A_\lambda$ given by $a + A_\mu \to a + A_\lambda$ (i. e. induced by the identity of A). φ_λ^λ will be the identity of A/A_λ. If $\alpha\colon (A, \mathbf{D}_A) \to (B, \mathbf{D}_B)$ is a morphism of the category $(\mathscr{A}, \mathbf{D})$, then it induces a map $\bar{\alpha}$ between the two inverse systems. For, to any pair $\lambda' \leqq \mu'$ there is a pair $\lambda \leqq \mu$

[8] For the properties of direct and inverse systems see [2].

such that the diagram

$$\begin{array}{ccc} A/A_\lambda & \leftarrow & A/A_\mu \\ \downarrow & & \downarrow \\ B/B_{\lambda'} & \leftarrow & B/B_{\mu'} \end{array} \tag{13}$$

is commutative. Dualizing the proof of Lemma 1 we obtain:

LEMMA 2. *The groups* $\mathrm{Hom}(A/A_\lambda, X|\mathbf{I})$ $(A_\lambda \in \mathbf{D}_A)$ *and the groups* $\mathrm{Ext}(A/A_\lambda, X)$ $(A_\lambda \in \mathbf{D}_A)$ *form direct systems where the homomorphisms are the corresponding induced maps between* Hom *and* Ext, *respectively. A map* σ *between the systems* $\{A/A/_\lambda\}$ *and* $\{B/B/_{\lambda'}\}$ *induces maps between the systems* $\{\mathrm{Hom}\,(B/B_{\lambda'}, X|\mathbf{I})\}$ *and* $\{\mathrm{Hom}\,(A/A_\lambda, X|\mathbf{I})\}$, *and between* $\{\mathrm{Ext}(B/B_{\lambda'}, X)\}$ *and* $\{\mathrm{Ext}\,(A/A_\lambda, X)\}$.

By making use of Lemmas 1 and 2 we are going to prove:

THEOREM 8. *For all categories* $(\mathcal{A}, \mathbf{I})$ *and* $(\mathcal{A}, \mathbf{D})$ *the following direct limit relations hold:*

$$\varinjlim_{A_\lambda \in \mathbf{I}_A} \mathrm{Hom}\,(X|\mathbf{D}, A_\lambda) = \mathrm{Hom}\,(X|\mathbf{D}, A|\mathbf{I})$$

and

$$\varinjlim_{A_\lambda \in \mathbf{D}_A} \mathrm{Hom}\,(A/A_\lambda, X|\mathbf{I}) = \mathrm{Hom}\,(A|\mathbf{D}, X|\mathbf{I}),$$

for any group X.

Let U_1, U_2 denote the indicated direct limits. We know that there exist unique homomorphisms

$$\pi_\lambda : \mathrm{Hom}\,(X|\mathbf{D}, A_\lambda) \to U_1 \quad \text{and} \quad \varphi_\lambda : \mathrm{Hom}\,(A/A_\lambda, X|\mathbf{I}) \to U_2$$

such that $\pi_\lambda^{\mu *}\, \pi_\mu = \pi_\lambda$ and $\varphi_\mu^{\lambda *}\, \varphi_\mu = \varphi_\lambda$. Here π_λ and φ_λ are necessarily monomorphisms, since the same holds for π_λ^μ and φ_μ^λ. We also have homomorphisms

$$\sigma_\lambda\colon \mathrm{Hom}(X|\mathbf{D}, A_\lambda) \to \mathrm{Hom}(X|\mathbf{D}, A|\mathbf{I}),$$

$$\tau_\lambda\colon \mathrm{Hom}(A/A_\lambda, X|\mathbf{I}) \to \mathrm{Hom}(A|\mathbf{D}, X|\mathbf{I})$$

induced by the inclusion map $A_\lambda \to A$ such that $\pi_\lambda^{\mu *}\, \sigma_\mu = \sigma_\lambda$ and $\varphi_\mu^{\lambda *}\, \tau_\mu = \tau_\lambda$. Therefore, by a known property of limits, there exist unique homomorphisms

$$\sigma : U_1 \to \mathrm{Hom}(X|\mathbf{D}, A|\mathbf{I}) \quad \text{and} \quad \tau : U_2 \to \mathrm{Hom}(A|\mathbf{D}, X|\mathbf{I})$$

such that $\pi_\lambda\, \sigma = \sigma_\lambda$ and $\varphi_\lambda\, \tau = \tau_\lambda$. From Theorems 6, 7 we know that σ_λ, τ_λ are monomorphisms. The same must be true for σ, τ, because if $\chi \in \mathrm{Ker}\,\sigma$ then there exist a λ and an $\eta_\lambda \in \mathrm{Hom}(X|\mathbf{D}, A_\lambda)$ satisfying $\eta_\lambda\, \pi_\lambda = \chi$ whence $\eta_\lambda\, \sigma_\lambda = \eta_\lambda\, \pi_\lambda\, \sigma = \chi\sigma = 0$ and $\eta_\lambda = 0$, $\chi = 0$; a similar inference applies to τ. Considering that the $\mathrm{Im}\,\sigma_\lambda$ together exhaust Hom $(X|\mathbf{D}, A|\mathbf{I})$ and the $\mathrm{Im}\,\tau_\lambda$ together exhaust $\mathrm{Hom}(A|\mathbf{D}, X|\mathbf{I})$, σ and τ are surjective. Consequently, they are isomorphisms, and Theorem 8 is completely proved.

The corresponding result on Ext does not hold, for in this case σ, τ need not be isomorphisms, merely epimorphisms. But let us define

$$\mathbf{I}\text{-Ext}(A, B) = \varinjlim_{B_\lambda \in \mathbf{I}_A} \operatorname{Ext}(A, B_\lambda)$$

and

$$\mathbf{D}\text{-Ext}(A, B) = \varinjlim_{A_\lambda \in \mathbf{D}_A} \operatorname{Ext}(A/A_\lambda, B) .$$

We then obtain, for any morphism β: $(B, \mathbf{I}_B) \to (C, \mathbf{I}_C)$ or α: $(C, \mathbf{D}_C) \to$ $\to (A, \mathbf{D}_A)$ the induced homomorphisms

$$\beta^*: \mathbf{I}\text{-Ext}(A, B) \to \mathbf{I}\text{-Ext}(A, C) \text{ and } \alpha^*: \mathbf{D}\text{-Ext}(A, B) \to \mathbf{D}\text{-Ext}(C, B)$$

whose detailed description may be left to the reader. It is also clear how **I**-Ext and **D**-Ext behave when their first and second variables, respectively, are subject to homomorphisms.

Note that there exist 'connecting homomorphisms' between certain Hom and Ext which are defined as maps between direct limits induced by the corresponding connecting homomorphisms of the members of the direct systems.

THEOREM 9. *If* $0 \to A \to B \to C \to 0$ *is an exact sequence, then for every* $(\mathscr{A}, \mathbf{D})$ *and* $(\mathscr{A}, \mathbf{I})$ *and for every group* X *the induced sequences*

$$\begin{aligned}(14)\qquad & 0 \to \operatorname{Hom}(X|\mathbf{D}, A) \to \operatorname{Hom}(X|\mathbf{D}, B) \to \operatorname{Hom}(X \mid \mathbf{D}, C) \to \\ & \to \mathbf{D}\text{-Ext}(X, A) \to \mathbf{D}\text{-Ext}(X, B) \to \mathbf{D}\text{-Ext}(X, C) \to 0 ,\end{aligned}$$

$$\begin{aligned}(15)\qquad & 0 \to \operatorname{Hom}(C, X|\mathbf{I}) \to \operatorname{Hom}(B, X|\mathbf{I}) \to \operatorname{Hom}(A, X|\mathbf{I}) \to \\ & \to \mathbf{I}\text{-Ext}(C, X) \to \mathbf{I}\text{-Ext}(B, X) \to \mathbf{I}\text{-Ext}(A, X) \to 0\end{aligned}$$

are exact. If $0 \to A \to B \to C \to 0$ *is* **I**-*exact, then*

$$\begin{aligned}(16)\qquad & 0 \to \operatorname{Hom}(X, A|\mathbf{I}) \to \operatorname{Hom}(X, B|\mathbf{I}) \to \operatorname{Hom}(X, C|\mathbf{I}) \to \\ & \to \mathbf{I}\text{-Ext}(X, A) \to \mathbf{I}\text{-Ext}(X, B) \to \mathbf{I}\text{-Ext}(X, C) \to 0\end{aligned}$$

is exact, while if $0 \to A \to B \to C \to 0$ *is* **D**-*exact, then*

$$\begin{aligned}(17)\qquad & 0 \to \operatorname{Hom}(C|\mathbf{D}, X) \to \operatorname{Hom}(B|\mathbf{D}, X) \to \operatorname{Hom}(A|\mathbf{D}, X) \to \\ & \to \mathbf{D}\text{-Ext}(C, X) \to \mathbf{D}\text{-Ext}(B, X) \to \mathbf{D}\text{-Ext}(A, X) \to 0\end{aligned}$$

is exact, for any X.

The sequence (14) may be viewed as the direct limit of exact sequences of the form (2) with X replaced by the factor groups X/X_λ, X_λ belonging to the dual ideal $\mathbf{D}_X$. Since φ^λ_μ: $X/X_\mu \to X/X_\lambda$ induces a map between the two exact sequences of type (2) (the arising diagram will be commutative), these maps will serve as the maps of the direct system of exact sequences.

The groups in (14) are the direct limits of the corresponding direct systems of groups, and since the direct limit of exact sequences is again exact, it follows that (14) is exact. The exactness of (15), (16) and (17) may be verified similarly.

In order to obtain a description of $\mathbf{I}$-Ext and $\mathbf{D}$-Ext in terms of factor sets, let us introduce the following notations. The set of all factor sets $f(a_1, a_2)$ of A into B such that the values of f belong to some $B^* \in \mathbf{I}_B$ (with B^* depending on f) will be denoted by $\mathrm{Fact}^{\mathbf{I}}(A, B)$, and those transformation sets $b(a_1) + b(a_2) - b(a_1 + a_2)$ of A into B for which there exists a subgroup $B^* \in \mathbf{I}_B$ such that $b(a) \in B^*$ for all $a \in A$, will be denoted by $\mathrm{Trans}^{\mathbf{I}}(A, B)$. Clearly, $\mathrm{Fact}^{\mathbf{I}}(A, B)$ contains $\mathrm{Trans}^{\mathbf{I}}(A, B)$ as a subset, and it is readily seen that they are both subgroups of $\mathrm{Fact}(A, B)$.

THEOREM 10. *The following natural isomorphism holds:*

$$\mathbf{I}\text{-Ext}(A, B) \cong \mathrm{Fact}^{\mathbf{I}}(A, B)/\mathrm{Trans}^{\mathbf{I}}(A, B).$$

Let $f(a_1, a_2)$ belong to $\mathrm{Fact}^{\mathbf{I}}(A, B)$ with values, say in $B_\lambda \in \mathbf{I}_B$. Then it induces some element of $\mathrm{Ext}(A, B_\lambda)$, and the application of the map $\pi_\lambda\colon \mathrm{Ext}(A, B_\lambda) \to \mathbf{I}\text{-Ext}(A, B)$ yields a map ϱ of $\mathrm{Fact}^{\mathbf{I}}(A, B)$ into $\mathbf{I}\text{-Ext}(A, B)$ which is evidently a homomorphism. To every element h of $\mathbf{I}\text{-Ext}(A, B)$ there is a subgroup $B_\lambda \in \mathbf{I}_B$ such that h is the image of some $h_\lambda \in \mathrm{Ext}(A, B_\lambda)$ under the map π_λ, as it follows from the general theory of direct limits. This implies that ϱ is surjective. We determine the kernel of ϱ. If f in $\mathrm{Fact}^{\mathbf{I}}(A, B)$ induces the zero of $\mathrm{Ext}(A, B_\lambda)$, then f is a transformation set with values in B_λ, i.e. f belongs to $\mathrm{Trans}^{\mathbf{I}}(A, B)$. Furthermore if $f' \in \mathrm{Ext}(A, B_\lambda)$, induced by f, is mapped upon 0 by ϱ then there exists a $\mu > \lambda$ such that $f' \pi_\lambda^\mu = 0$, i.e. f' is mapped upon 0 by the homomorphism $\mathrm{Ext}(A, B_\lambda) \to \mathrm{Ext}(A, B_\mu)$, which shows that f' is a transformation set with values in B_μ, and so $f \in \mathrm{Trans}^{\mathbf{I}}(A, B)$. Since every element of $\mathrm{Trans}^{\mathbf{I}}(A, B)$ clearly belongs to $\mathrm{Ker}\,\varrho$, the stated isomorphism follows which is evidently natural.

To obtain the corresponding result for $\mathbf{D}$-Ext, let us analogously define $\mathrm{Fact}^{\mathbf{D}}(A, B)$ as the set of all factor sets $f(a_1, a_2)$ of A into B for which there exists some A^* in $\mathbf{D}_A$(with A^* depending on f) such that $f(a_1, a_2) = f(a_1', a_2')$ whenever $a_i \equiv a_i' (\mathrm{mod}\ A^*)$, and $\mathrm{Trans}^{\mathbf{D}}(A, B)$ as the set of all transformation sets $b(a_1) + b(a_2) - b(a_1 + a_2)$ of A into B for which there exists some $A^* \in \mathbf{D}_A$ such that $b(a) = b(a')$ for $a \equiv a' (\mathrm{mod}\ A^*)$. These are again subgroups of $\mathrm{Fact}(A, B)$ and we have:

THEOREM 11. *The following natural isomorphism holds:*

$$\mathbf{D}\text{-Ext}(A, B) \cong \mathrm{Fact}^{\mathbf{D}}(A, B)/\mathrm{Trans}^{\mathbf{D}}(A, B).$$

The proof is entirely similar to that of Theorem 10.

To conclude this section let us note that $\mathbf{I}$-Ext and $\mathbf{D}$-Ext can be combined into a single Ext whose detailed discussion, however, is not intended here.

§ 3. Certain subgroups of Ext

Starting with the categories $(\mathscr{A}, \mathbf{I})$ and $(\mathscr{A}, \mathbf{D})$ we introduce four main kinds of groups of extensions which are all subgroups of Ext.

By an extension of B by A (in the category $\mathscr{A}$) an exact sequence

$$0 \to B \xrightarrow{\mu} G \xrightarrow{\nu} A \to 0 \tag{18}$$

is meant and the equivalence of two extensions is defined in the usual way. If A^* is a subgroup of A, (18) induces an extension of B by A^*

$$0 \to B \to G^* \xrightarrow{\nu^*} A^* \to 0 \tag{19}$$

where $G^* = A^* \nu^{-1}$ and ν^* is the restriction of ν to G^*. Similarly, if B^* is a subgroup of B, (18) induces the extension

$$0 \to B/B^* \xrightarrow{\mu^*} G_* \to A \to 0 \tag{20}$$

of B/B^* by A, where $G_* = G/B^*\mu$ and μ^* is derived from μ by means of the natural homomorphism $B \to B/B^*$. We shall refer to these induced extensions in our following definitions. Note that if (19) splits then so does every sequence of the same type obtained from an extension equivalent to (18). A similar remark applies to (20).

(a) Let $\operatorname{Ext}(A|\mathbf{I}, B)$ consist of all extensions (18) of B by A for which the induced extension (19) splits for *all* $A^* \in \mathbf{I}_A$;

(b) let $\operatorname{Ext}(A, B|\mathbf{I})$ be the set of all extensions (18) of B by A for which the induced extension (20) splits for *some* $B^* \in \mathbf{I}_B$;

(c) let $\operatorname{Ext}(A|\mathbf{D}, B)$ denote the set of all extensions (18) such that the induced extension (19) splits for *some* $A^* \in \mathbf{D}_A$;

(d) finally let $\operatorname{Ext}(A, B|\mathbf{D})$ consist of all extensions (18) such that the induced extension (20) splits for *all* $B^* \in \mathbf{D}_B$.

If we combine any one of (a), (c) with one of (b), (d), we obtain the definitions of four other Exts, e.g. $\operatorname{Ext}(A|\mathbf{I}, B|\mathbf{J})$. From the definitions it is clear that if $\mathbf{I} \subseteq \mathbf{J}$ and $\mathbf{D} \subseteq \mathbf{E}$ then we have the following inclusions :

$$\operatorname{Ext}(A|\mathbf{I}, B) \supseteq \operatorname{Ext}(A|\mathbf{J}, B), \qquad \operatorname{Ext}(A, B|\mathbf{I}) \subseteq \operatorname{Ext}(A, B|\mathbf{J}),$$

$$\operatorname{Ext}(A|\mathbf{D}, B) \subseteq \operatorname{Ext}(A|\mathbf{E}, B), \qquad \operatorname{Ext}(A, B|\mathbf{D}) \supseteq \operatorname{Ext}(A, B|\mathbf{E}).$$

Next we verify:

THEOREM 12. *The subsets of* $\operatorname{Ext}(A, B)$ *defined above in* (a)—(d) *are subgroups of* $\operatorname{Ext}(A, B)$ *and they satisfy*

$$\begin{aligned}
\operatorname{Ext}(A|\mathbf{I}, B) \cap \operatorname{Ext}(A, B|\mathbf{J}) &= \operatorname{Ext}(A|\mathbf{I}, B|\mathbf{J}),\\
\operatorname{Ext}(A|\mathbf{I}, B) \cap \operatorname{Ext}(A, B|\mathbf{D}) &= \operatorname{Ext}(A|\mathbf{I}, B|\mathbf{D}),\\
\operatorname{Ext}(A|\mathbf{D}, B) \cap \operatorname{Ext}(A, B|\mathbf{I}) &= \operatorname{Ext}(A|\mathbf{D}, B|\mathbf{I}),\\
\operatorname{Ext}(A|\mathbf{D}, B) \cap \operatorname{Ext}(A, B|\mathbf{E}) &= \operatorname{Ext}(A|\mathbf{D}, B|\mathbf{E}).
\end{aligned}$$

By virtue of the definitions it is sufficient to establish the first assertion. All of the sets defined in (a)—(d) contain the direct sum of B and A, thus they are not void. Representing (18) by a factor set $g(a_1, a_2)$ where $a_i \in A$ and $g(a_1, a_2) \in B$, (19) splits if and only if $g(a_1, a_2)$ restricted to $a_i \in A^*$ is a transformation set. Since the difference of extensions corresponds to the difference of the corresponding factor sets, it is evident that if two extensions (18) of B by A split on passing to the induced extensions (19) for some $A^* \subseteq A$, then so does their difference for the same A^*. This implies that (a) is a subgroup. If, in addition, we note that if (19) splits for some $A^* \subseteq A$ and if $A' \subseteq A^*$, then (19) will split for A', too, then the dual ideal property of $\mathbf{D}_A$ proves that (c) is likewise a subgroup. The splitting of (20) means that $g(a_1, a_2)$ is mod B^* a transformation set. Hence the subgroup character of (d) is obvious. Now if $g(a_1, a_2)$ is a transformation set mod B^* and if $B' \supseteq B^*$, then it is one mod B' too, whence the ideal property of $\mathbf{I}_B$ guarantees that the difference of two elements of $\mathrm{Ext}(A, B|\mathbf{I})$ is again an element of the same set, i.e. (b) is a subgroup too. The proof of Theorem 12 is thereby completed.

We turn to the discussion of induced homomorphisms between the subgroups of Ext now introduced. Given the morphisms

$$\varphi : (X, \mathbf{I}_X) \to (Y, \mathbf{I}_Y) \quad \text{and} \quad \psi : (X, \mathbf{D}_X) \to (Y, \mathbf{D}_Y) \tag{21}$$

of the categories $(\mathscr{A}, \mathbf{I})$ and $(\mathscr{A}, \mathbf{D})$, respectively, there exist induced homomorphisms

$$\begin{aligned} \mathrm{Ext}\,(\iota, \varphi) &: \mathrm{Ext}\,(A|\ \ , X|\mathbf{I}) \to \mathrm{Ext}\,(A|\ \ , Y|\mathbf{I}) \\ \mathrm{Ext}\,(\iota, \psi) &: \mathrm{Ext}\,(A|\ \ , X|\mathbf{D}) \to \mathrm{Ext}\,(A|\ \ , Y|\mathbf{D}) \end{aligned} \tag{22}$$

and

$$\begin{aligned} \mathrm{Ext}\,(\varphi, \iota) &: \mathrm{Ext}\,(Y|\mathbf{I}, A|\ \ ) \to \mathrm{Ext}\,(X|\mathbf{I}, A|\ \ ) \\ \mathrm{Ext}\,(\psi, \iota) &: \mathrm{Ext}\,(Y|\mathbf{D}, A|\ \ ) \to \mathrm{Ext}\,(X|\mathbf{D}, A|\ \ ) \end{aligned} \tag{23}$$

where on the blank places after A any $\mathbf{J}$ or any $\mathbf{E}$ may stand, and the homomorphisms (22), (23) act in the following way:

$$g(a_1, a_2) \to g(a_1, a_2)\,\varphi \qquad \text{resp.} \quad g(a_1, a_2)\,\psi \quad (a_i \in A)\,, \tag{24}$$

$$g(y_1, y_2) \to g\,(x_1\,\varphi, x_2\,\varphi) \quad \text{resp.} \quad g(x_1\,\psi, x_2\,\psi)\,(x_i \in X, y_i \in Y). \tag{25}$$

We have to verify that these are good homomorphisms indeed, i.e. they map the groups on the left members into the groups on the right members of (22), (23).

Since the equality of $g(a_1, a_2)$ to a transformation set, under the restriction of the a_i to a subgroup of A, is a property which is preserved under the mappings (22), we may focus our attention on what happens modulo subgroups of X and Y. If $g(a_1, a_2)$ represents an element of $\mathrm{Ext}(A|\ \ , X|\mathbf{I})$

or $\operatorname{Ext}(A| \ , X|\mathbf{D})$, then it is a transformation set modulo some $X^* \in \mathbf{I}_X$ or any $X^* \in \mathbf{D}_X$. Hence $g(a_1, a_2)\varphi$ is a transformation set mod $X^*\varphi = Y^* \in \mathbf{I}_Y$. Considering $g(a_1, a_2)\psi$ modulo an arbitrary $Y^* \in \mathbf{D}_Y$, we select to this Y^* an $X^* \in \mathbf{D}_X$ such that $X^*\psi \subseteq Y^*$. Since $g(a_1, a_2)\psi$ is obviously a transformation set mod $X^*\psi$, it is one mod Y^* too. Therefore $g(a_1, a_2)\varphi \in \operatorname{Ext}(A| \ , Y|\mathbf{I})$ and $g(a_1, a_2)\psi \in \operatorname{Ext}(A| \ , Y|\mathbf{D})$.

In dealing with (23), we may again leave the ideal or the dual ideal of A out of consideration. If $g(y_1, y_2)$ represents an element of $\operatorname{Ext}(Y|\mathbf{I}, A| \ )$ or $\operatorname{Ext}(Y|\mathbf{D}, A| \ )$, then it is a transformation set if the y_i are restricted to range over an arbitrary $Y^* \in \mathbf{I}_Y$ or a suitable $Y^* \in \mathbf{D}_Y$. Since $\mathbf{I}_X\varphi \subseteq \mathbf{I}_Y$, $g(x_1\varphi, x_2\varphi)$ is a transformation set if the x_i are restricted to any $X^* \in \mathbf{I}_X$. Since to Y^* there exists an $X^* \in \mathbf{D}_X$ with $X^*\psi \subseteq Y^*$, $g(x_1\psi, x_2\psi)$ is a transformation set if the x_i are restricted to X^*. This proves that $g(x_1\varphi, x_2\varphi)$ belongs to $\operatorname{Ext}(X|\mathbf{I}, A| \ )$ and $g(x_1\psi, x_2\psi)$ belongs to $\operatorname{Ext}(X|\mathbf{D}, A| \ )$.

Exact sequences for the subgroups of Ext now introduced can be established under certain conditions on the exact sequence we start with. Before considering these exact sequences we turn our attention to the connecting homomorphisms in order to prove immediately exact sequences like (2) and (3).

The next result shows what can be said of the connecting homomorphisms

$$\gamma : \eta \to g(x_1\eta, x_2\eta) \quad \text{and} \quad \delta : \chi \to g(c_1, c_2)\chi$$

of (2) and (3) where $g(c_1, c_2)$ is a factor set representing the exact sequence

$$0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0. \tag{26}$$

THEOREM 13. (i) *γ maps* $\operatorname{Hom}(X|\mathbf{D}, C)$ *into* $\operatorname{Ext}(X|\mathbf{D}, A)$, *and δ maps* $\operatorname{Hom}(A, X|\mathbf{I})$ *into* $\operatorname{Ext}(C, X|\mathbf{I})$.

(ii) *If* (26) *represents an element of* $\operatorname{Ext}(C, A|\mathbf{I})$, *then* $\operatorname{Im}\gamma \subseteq \operatorname{Ext}(X, A|\mathbf{I})$, *and if it represents an element of* $\operatorname{Ext}(C, A|\mathbf{D})$ *then* $\operatorname{Im}\gamma \subseteq \operatorname{Ext}(X, A|\mathbf{D})$.

(iii) *If* (26) *belongs to* $\operatorname{Ext}(C|\mathbf{I}, A)$ *or to* $\operatorname{Ext}(C|\mathbf{D}, A)$, *then* $\operatorname{Im}\gamma$ *is included in* $\operatorname{Ext}(C|\mathbf{I}, X)$ *or in* $\operatorname{Ext}(C|\mathbf{D}, X)$.

(iv) *If* (26) *represents an element of* $\operatorname{Ext}(C|\mathbf{I}, A)$, *then γ sends* $\operatorname{Hom}(X, C|\mathbf{I})$ *into* 0, *and if it represents an element of* $\operatorname{Ext}(C, A|\mathbf{D})$, *then δ sends* $\operatorname{Hom}(A|\mathbf{D}, X)$ *into* 0.

To prove (i), let $\operatorname{Ker}\eta = X^* \in \mathbf{D}_X$. Then $g(x_1\eta, x_2\eta)$ is obviously a transformation set if the x_i are restricted to X^*. If $\operatorname{Im}\chi = X^* \in \mathbf{I}_X$, then $g(c_1, c_2)\chi$ is evidently a transformation set modulo this X^*.

Under the hypothesis of (ii), $g(x_1\eta, x_2\eta)$ is a transformation set modulo some $A^* \in \mathbf{I}_A$ or modulo all $A^* \in \mathbf{D}_A$.

Under (iii), $g(c_1, c_2)\chi$ is a transformation set if the c_i are subject to the restriction to range over an arbitrary $C^* \in \mathbf{I}_C$ or over some $C^* \in \mathbf{D}_C$.

The assumptions of (iv) guarantee that $g(x_1\eta, x_2\eta)$ and $g(c_1, c_2)\chi$ are transformation sets.

We obtain from part (iv) of the preceding theorem and from Theorems 6, 7:

THEOREM 14. *If* $0 \to A \to B \to C \to 0$ *is an* **I**-*exact sequence such that the corresponding extension of A by C belongs to* $\mathrm{Ext}(C|\mathbf{I}, A)$, *then the sequence*

$$0 \to \mathrm{Hom}(X|\mathbf{D}, A|\mathbf{I}) \to \mathrm{Hom}(X|\mathbf{D}, B|\mathbf{I}) \to \mathrm{Hom}(X|\mathbf{D}, C|\mathbf{I}) \to 0$$

is exact for every group X and category $(\mathscr{A}, \mathbf{D})$. *If we replace the assumption of* **I**-*exactness by* **D**-*exactness and* $\mathrm{Ext}(C|\mathbf{I}, A)$ *by* $\mathrm{Ext}(C, A|\mathbf{D})$, *then for every group X and category* $(\mathscr{A}, \mathbf{I})$ *the sequence*

$$0 \to \mathrm{Hom}(C|\mathbf{D}, X|\mathbf{I}) \to \mathrm{Hom}(B|\mathbf{D}, X|\mathbf{I}) \to \mathrm{Hom}(A|\mathbf{D}, X|\mathbf{I}) \to 0$$

is exact.

Though we shall not need the following result, we shall prove it since it is of some interest.

THEOREM 15. *If the extension* (26) *of A by C belongs to* $\mathrm{Ext}(C|\mathbf{I}, A)$, *then γ maps* $\mathrm{Hom}_{\mathbf{I}}(X, C)$ *into* $\mathrm{Ext}(X|\mathbf{I}, A)$, *while if it belongs to* $\mathrm{Ext}(C, A|\mathbf{D})$, *then δ maps* $\mathrm{Hom}_{\mathbf{D}}(A, X)$ *into* $\mathrm{Ext}(C, X|\mathbf{D})$.

Under the first hypothesis $g(c_1, c_2)$ is a transformation set whenever the c_i range over an arbitrary $C^* \in \mathbf{I}_C$. For any $X^* \in \mathbf{I}_X$ the image $X^*\eta$ belongs to $\mathbf{I}_C$ whence $g(x_1\eta, x_2\eta)$ is a transformation set if the x_i are restricted to any $X^* \in \mathbf{I}_X$. Under the second assumption, $g(c_1, c_2)$ is a transformation set modulo each $A^* \in \mathbf{D}_A$. If X^* is given in $\mathbf{D}_X$, there is an $A^* \in \mathbf{D}_A$ such that $A^*\chi \subseteq X^*$. Hence $g(c_1, c_2)\chi$ is a transformation set mod $A^*\chi$, and so mod X^* too. This completes the proof.

Now we are ready to prove the main results of this section. γ and δ will denote throughout the connecting homomorphisms.

THEOREM 16. *Let* (26) *be a* **D**-*exact sequence representing an element of* $\mathrm{Ext}(C, A|\mathbf{D})$. *Then the following induced sequence is exact:*

$$0 \to \mathrm{Hom}(X, A) \to \mathrm{Hom}(X, B) \to \mathrm{Hom}(X, C) \xrightarrow{\gamma} \tag{27}$$
$$\to \mathrm{Ext}(X, A|\mathbf{D}) \xrightarrow{\alpha^*} \mathrm{Ext}(X, B|\mathbf{D}) \xrightarrow{\beta^*} \mathrm{Ext}(X, C|\mathbf{D}) \to 0.$$

It has already been shown, that the indicated homomorphisms are in fact maps between the indicated groups. It is clear that the composite of any two consecutive maps is zero. The exactness at $\mathrm{Hom}(X, C)$ and at $\mathrm{Ext}(X, A|\mathbf{D})$ is a simple consequence of that of (2). Thus it remained to verify the exactness at the two last Exts.

Let the factor set $f(x_1, x_2)$ represent an element of $\mathrm{Ext}(X, B|\mathbf{D})$ in the kernel of β^*. It is then a transformation set modulo each $B^* \in \mathbf{D}_B$ and $f(x_1, x_2)\beta$ is also one. By (2), there exists an $h(x_1, x_2)$ in $\mathrm{Ext}(X, A)$ such that

$h(x_1, x_2)\alpha$ is equivalent to $f(x_1, x_2)$. There is no loss of generality in assuming that $h(x_1, x_2)\alpha = f(x_1, x_2)$. Now let us consider $h\,\alpha$ mod $A^*\,\alpha$ for some $A^* \in \mathbf{D}_A$. By the $\mathbf{D}$-exactness of (26) there exists a $B^* \in \mathbf{D}_B$ such that $B^* \cap \cap A\,\alpha = A^*\,\alpha$. Because of the other assumption on (26) we may write $B/A^*\,\alpha = C \oplus A\,\alpha/A^*\,\alpha$. It follows that there exists a function $x \to b(x)$ of X into B satisfying $h(x_1, x_2)\alpha = b(x_1) + b(x_2) - b(x_1 + x_2) - b^*(x_1, x_2)$ with $b^*(x_1, x_2) \in B^*$, and thus by the given direct decomposition we can write

$$h(x_1, x_2)\,\alpha \equiv c(x_1) + c(x_2) - c(x_1 + x_2) - c^*(x_1, x_2) + \\ + a(x_1) + a(x_2) - a(x_1 + x_2) - a^*(x_1, x_2) \bmod A^*\,\alpha$$

with obvious notation. In view of $B^* \cap A\,\alpha \equiv 0 \bmod A\,\alpha$, $C^* = B^*\beta \cong B^*/A^*\alpha$ is isomorphic to its C-component and we have a homomorphism $\chi\colon c^* \to a^*$ of C^* into $A\,\alpha/A^*\,\alpha$. We also have $c^*(x_1, x_2) = c(x_1) + c(x_2) - c(x_1 + x_2)$ which implies that under the homomorphism $\mathrm{Ext}(X, B^*) \to \mathrm{Ext}(X, C^*)$ the factor set $b^*(x_1, x_2)$ of X into B^* is mapped upon a transformation set, consequently, $b^*(x_1, x_2)$ is equivalent to $k(x_1, x_2)\,\alpha$ for some $k(x_1, x_2)$ in $\mathrm{Ext}(X, A^*)$: $b^*(x_1, x_2) = k(x_1, x_2)\,\alpha + b^*(x_1) + b^*(x_2) - b^*(x_1 + x_2)$ for a function $x \to b^*(x)$ of X into B^*. Hence $c^*(x_1, x_2) = c^*(x_1) + c^*(x_2) - c^*(x_1 + x_2)$ with $c^* \in C^*$, and an application of χ yields $a^*(x_1, x_2) = c^*(x_1)\,\chi + c^*(x_2)\,\chi - c^*(x_1 + x_2)\,\chi$. Consequently, $h(x_1, x_2)\,\alpha = d(x_1) + d(x_2) - d(x_1 + x_2) \bmod A^*\,\alpha$ with $d(x) = = a(x) - c^*(x)\,\chi \in A\,\alpha/A^*\alpha$ and we conclude that h is in fact a transformation set mod A^*. This establishes the exactness at $\mathrm{Ext}(X, B|\mathbf{D})$.

Finally, we prove that β^* is an epimorphism. From the exactness of (2) we infer that, given an element of $\mathrm{Ext}(X, C|\mathbf{D})$, say $f(x_1, x_2)$, there exists a factor set $h(x_1, x_2)$ representing an element of $\mathrm{Ext}(X, B)$ such that $h(x_1, x_2)\beta$ is equivalent to, moreover, equal to $f(x_1, x_2)$. Given a $B^* \in \mathbf{D}_B$, we have $C^* = B^*\,\beta \in \mathbf{D}_C$. Now $f(x_1, x_2)$ is a transformation set modulo C^*, that is, $f(x_1, x_2) \equiv c(x_1) + c(x_2) - c(x_1 + x_2) \bmod C^*$ for certain elements $c(x) \in C$ $(x \in X)$. Hence $h(x_1, x_2) \equiv b(x_1) + b(x_2) - b(x_1 + x_2) \bmod B^*$ for arbitrary inverse images $b(x)$ of $c(x)$ under β. Thus h represents an element of $\mathrm{Ext}(X, B|\mathbf{D})$ and β in (27) is surjective. This completes the proof of the theorem.

THEOREM 17. *Let* (26) *be an* $\mathbf{I}$*-exact sequence such that it represents an element of* $\mathrm{Ext}(C|\mathbf{I}, A)$. *Then the induced sequence*

$$(28)\qquad 0 \to \mathrm{Hom}\,(C, X) \to \mathrm{Hom}\,(B, X) \to \mathrm{Hom}\,(A, X) \xrightarrow{\delta} \\ \to \mathrm{Ext}\,(C|\mathbf{I}, X) \xrightarrow{\beta^*} \mathrm{Ext}\,(B|\mathbf{I}, X) \xrightarrow{\alpha^*} \mathrm{Ext}\,(A|\mathbf{I}, X) \to 0$$

is exact.

Analogous to the proof of the preceding theorem we conclude that it suffices to verify the exactness at the two last Exts.

Let $f(b_1, b_2)$ represent an element of $\mathrm{Ext}(B|\mathbf{I}, X)$ in the kernel of α^*, i.e. $f(a_1\alpha, a_2\alpha)$ is a transformation set $(b_i \in B, a_i \in A)$, and $f(b_1, b_2)$ is a transformation set when restricted to any $B^* \in \mathbf{I}_B$. By the exactness of (3) there exists an $h(c_1, c_2)$ in $\mathrm{Ext}(C, X)$ such that $f(b_1, b_2)$ is equivalent to, or moreover equal to $h(b_1\beta, b_2\beta)$. Choose some $C^* \in \mathbf{I}_C$, and let $B^* \in \mathbf{I}_B$ be such that $C^* = B^*\beta$. By hypothesis there is a function $b \to x(b)$ of B^* into X such that $h(b_1\beta, b_2\beta) = x(b_1) + x(b_2) - x(b_1 + b_2)$ for $b_i \in B^*$. Take any direct decomposition $C^*\beta^{-1} = A\alpha \oplus C'$ and write $b = a\alpha + b' \in B^*$. Then define $y(b) = x(b')$ which is evidently unique and $y(b_1 + b_2) = x(b_1' + b_2')$ holds. We have $h(b_1\beta, b_2\beta) = h(b_1'\beta, b_2'\beta) = x(b_1') + x(b_2') - x(b_1' + b_2') = y(b_1) + y(b_2) - y(b_1 + b_2)$. The function $b \to y(b)$ assumes the same values in the cosets of B^* mod $A\alpha$ and therefore induces a function $c \to y(c)$ for $c \in C^*$. Thus $h(c_1, c_2) = y(c_1) + y(c_2) - y(c_1 + c_2)$ which shows that f belongs to $\mathrm{Im}\,\beta^*$.

In order to show that α is surjective, let us take an $f(a_1, a_2)$ representing an element of $\mathrm{Ext}(A|\mathbf{I}, X)$; then f is a transformation set when restricted to any $A^* \in \mathbf{I}_A$. If B^* is arbitrary in $\mathbf{I}_B$, then $A^*\alpha = B^* \cap A\alpha$ for some $A^* \in \mathbf{I}_A$. The given extension f of X by A and the splitting extension of X by B^* yield an extension $G(B^*)$ of X by the subgroup of B generated by $A\alpha$ and B^*. If B^* runs over all subgroups in $\mathbf{I}_B$, these $G(B^*)$ give rise to an extension of X by some subgroup of B.[9] This extension can be continued to an extension of X by B. This is clearly splitting when restricted to any $B^* \in \mathbf{I}_B$. This extension (considered as an element of $\mathrm{Ext}(B, X)$) is mapped by α just onto f. This completes the proof of Theorem 17.

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[9] This can be shown e.g. by using the direct system of exact sequences $0 \to X \to G(B^*) \to \{A, B^*\} \to 0$.

ADDITIVE GROUPS OF INTEGER-VALUED FUNCTIONS OVER TOPOLOGICAL SPACES

By

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1. Two continuous mappings (functions) f and g of a topological space X into the set of integers I can be added in the natural way, i.e. argument-wise:

$$(f+g)(x) = f(x) + g(x) \quad (x \in X).$$

Hence, every set of such mappings generates an Abelian group of (continuous) mappings and it is natural to ask for the structure of such a group.

The *main conjecture* states that for every X, and for every such group of *bounded* continuous functions, this group will be *free Abelian*.

If one also takes unbounded functions into account, the generated groups are not necessarily free Abelian. This follows already from a well-known result of Baer which states that the unrestricted direct sum of free Abelian groups is not completely decomposable ($X = I$ in this case).

In this paper we will first reduce the problem of general topological spaces to special ones. In Section 2 we will show that it is not difficult to solve the conjecture in the affirmative in the case of countable groups. In Section 3 we will state a number of cases in which the conjecture also holds.

In the sequel all maps $f : X \to I$ will be bounded and continuous. For any such f the set fX consists of a finite number of points and the pre-images determine a decomposition D of X into a finite number of disjoint clopen (open and closed) sets. If we assign to each of these its value in I (i.e. the number of I on which it is mapped by f) we obtain a complete description of f by means of this 'numbered' decomposition ND into 'numbered' clopen subsets of X. Conversely, to every such numbered decomposition ND corresponds exactly one such f.

The intersection of all clopen subsets containing a point $x \in X$ determines the *quasi-component* Q_x of x in X. As is well known, X can be decomposed in its (mutually disjoint) quasi-components, each quasi-component being a closed subset of X. The natural map φ is the map which maps every $x \in X$ onto Q_x.

The *quasi-component space* $Q(X)$ of X is defined as the topological space for which the quasi-components are the points and which has for a base all sets $\varphi(C) \subset Q(X)$, where C is an arbitrary clopen subset of X. In this way $Q(X)$ becomes a completely regular topological space (as can easily be seen) and every f under consideration determines an $f^*: Q(X) \to I$ in a natural way, and conversely. Groups $\{f\}$ are isomorphic to groups $\{f^*\}$, and conversely. Hence, we have to study only the groups for the spaces $Q(X)$. Since $Q(X)$ is completely regular, we can restrict our field of interest even more. Indeed, take the maximal zero-dimensional Hausdorff compactification $\gamma\, Q(X)$ of $Q(X)$ (this exists and coincides in most cases with the Čech-Stone compactification $\beta\, Q(X)$ of $Q(X)$), then every f^*: $Q(X) \to I$ can be extended uniquely to a map $\tilde{f}^*$: $\gamma\, Q(X) \to I$, since f^* is bounded.

Conversely, such an $\tilde{f}^*$ determines f^* uniquely, since $Q(X)$ is dense in $\gamma\, Q(X)$. So we have shown the following: every group of functions under consideration, say G: $X \to I$ induces an isomorphic group G^*: $\gamma\, Q(X) \to I$; conversely, G is determined by, and isomorphic to G^*.

So, *instead of studying the groups of functions of X into I, one only has to study these for compact, zero-dimensional Hausdorff spaces.*

Another simple but useful general remark, let us call it the *continuity remark*, is the following. If ψ: $X \to Y$ is a continuous map of X onto Y and G: $Y \to I$ is one of our groups under consideration, then the set of maps $\{g\,\psi\}_{g\in G}$ determines a group G': $X \to I$, and G and G' are isomorphic.

2. We shall give two proofs, a mainly topological one and a mainly group-theoretical one of the following result.

THEOREM. *Every countable group of continuous bounded integer-valued functions over a topological space is free Abelian.*

In the topological proof we use the following lemma.

LEMMA. *The full group (of all functions) G: $C \to I$, where C denotes the Cantor set (i.e. the discontinuum of Cantor) is free Abelian.*

PROOF. C is a product of a countable number of doublets (a doublet is a pair of isolated points). To a doublet or a finite set of doublets corresponds a natural decomposition of C into disjoint clopen sets. Enumerate the doublets. To the first doublet and its decomposition D_1 one lets correspond two base elements of G, namely $1, 1$ and $0, 1$; that is $1, 1$ is the function identical to 1, and $0, 1$ is the function equal to 0 in the first and equal to 1 in the second clopen set of the decomposition D_1.

In matrix form

$$G_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

After having defined G_{n-1} over the natural decomposition D_{n-1} of C in

2^{n-1} disjoint clopen sets corresponding to the first $(n-1)$ doublets, we define G_n by

$$G_n = \begin{pmatrix} G_{n-1} & G_{n-1} \\ 0 & G_{n-1} \end{pmatrix}.$$

The 2^n rows in this matrix define 2^n base elements (these include all previously defined base elements corresponding to G_i for $i < n$).

Now we let n tend to infinity and show that the set of all base elements B thus obtained, is a free base of G. Firstly, every finite system of base elements is a set of certain rows in some G_n and these constitute a free Abelian group of finite rank, since

$$\det G_n = 1 .$$

Secondly, every $g \in G$ is uniquely determined, as can easily be seen, by its values over a certain decomposition D_m of C (i.e. to each decomposition element is assigned some fixed integer), so g is an element of the group generated by the rows of G_m. Hence, G is the restricted direct sum of the base elements, as mentioned, so it is free Abelian.

The group-theoretical proof uses the following simple *criterion of Pontrjagin*: A countable Abelian group is free Abelian if and only if it is torsion-free and every subgroup of finite rank contains finitely many generators (so must be free).

Topological proof of theorem. Since the given group G is countable, the number of corresponding decompositions D is also countable. The intersection of all those D which contain a point x defines a closed subset—let us call it the G-component of x—of the space. Hence the space is decomposed into mutually disjoint, closed G-components. The quotient space over the G-components becomes a zero-dimensional regular space which has a countable base, since the total number of clopen sets corresponding to the D's is countable. Hence, S is separable metrizable and G can be thought of as a group operating over S. Now we can apply an extension theorem by McDowell and the author [1], which yields the result that each $g \in G$ can be extended continuously to a suitable compact metrizable compactification $\overline{S}$ of S. Since S is dense in $\overline{S}$, the corresponding group $\overline{G}$ is isomorphic to G.

There exists, as is well known, a continuous map $\varphi: C \to \overline{S}$ and by the continuity remark, it follows that G is a subgroup of the group mentioned in the lemma, so free itself.

Group-theoretical proof of theorem. We apply Pontrjagin's criterion. The given group G is clearly torsion-free. Consider some subgroup of finite rank H of G. Take a maximal, independent set of elements in H and consider the ND's corresponding to these elements. The finite intersection of all sets D occurring in any of the ND's gives us a decomposition of the space in finitely many clopen subsets S_i. Consider the functions f_i each f_i being 1 on S_i and 0 elsewhere. These f_i generate a free group which contains H as a subgroup. So H is free, q.e.d.

COROLLARY. *The group of all continuous integer-valued functions over a compact metrizable space is free Abelian.*

Indeed, this group is countable.

3. We can generalize the proof of the lemma to a proof which holds for any generalized discontinuum D^m (product of any number m of doublets) and this shows that the groups under consideration are free if the space is D^m or a continuous image of it, i.e. a *dyadic space*.

Specker [2] proved, *using the continuum hypothesis*, that the full group $G: I \to I$, so the group of all bounded integer-valued sequences under argument-wise addition, is free.

From this it follows that the *full group*

$$G: X \to I$$

for any separable space X is free, if we assume the continuum hypothesis. Indeed, take a countable subset M in X. Take some "onto" map $\varphi: I \to M$, then the full group $G_M: M \to I$ is free because of the continuity remark. On the other hand, G is clearly a subgroup of G_M, so G is free.

So our conjecture also holds for dyadic spaces and (using the continuum hypothesis) for separable spaces. A proof of the conjecture in general still presents difficulties, although it may well be possible that the use of inverse limit techniques might render a proof. Actually, one has only to apply these—because of the continuity remark and the properties of Čech-Stone compactifications—for the case of such a compactification of a discrete space.

The problems in this paper can be generalized in replacing I by a (topological) Abelian group whatsoever.

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ON THE STRUCTURE OF ABELIAN p-GROUPS

By

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When G is a reduced Abelian p-group of length λ, we consider the following descending sequence of subgroups of the socle S_0 of G

$$\{S_\alpha\} : S_0 \supseteq S_1 \supseteq S_2 \supseteq \ldots \supseteq S_\alpha \supseteq \ldots \supset S_\lambda = 0\,,$$

where $S_\alpha = S_0 \cap p^\alpha G$ for every ordinal $\alpha (0 \leqq \alpha \leqq \lambda)$. (For most terminologies for Abelian groups, we refer to Fuchs [1].) For any limit ordinal $\varrho \leqq \lambda$, it holds $S_\varrho = \bigcap_{\alpha<\varrho} S_\alpha$. For any ordinal $\alpha < \lambda$, there exists a finite ordinal n such that $S_{\alpha+n} \neq S_{\alpha+n+1}$.

Next, we take an ascending sequence of subgroups of S_0

$$\{P_\alpha\} : 0 = P_0 \subseteq P_1 \subseteq P_2 \subseteq \ldots \subseteq P_\alpha \subseteq \ldots \subseteq P_\lambda = S_0$$

such that, for any ordinal $\alpha \leqq \lambda$, it holds

$$S_0 = P_\alpha + S_\alpha .$$

Put

$$P_{\alpha+1} \cap S_\alpha = C_\alpha$$

for every $\alpha < \lambda$. Then we have, for any $\alpha < \lambda$,

$$S_\alpha = C_\alpha + S_{\alpha+1}\,,$$

and

(1) $$S_0 = P_\alpha + C_\alpha + S_{\alpha+1} .$$

Let the complete direct sum of the groups C_α $(\alpha < \lambda)$ be denoted by V. Any element of V is written as a vector

$$\langle c_0, c_1, \ldots, c_\alpha, \ldots \rangle \quad (c_\alpha \leq C_\alpha).$$

We now obtain a mapping from S_0 into V. In fact, take any element u of S_0, and, for any $\alpha < \lambda$, express u in view of the decomposition (1) as

$$u = p_\alpha + c_\alpha + s_{\alpha+1} \quad (p_\alpha \in P_\alpha, c_\alpha \in C_\alpha, s_{\alpha+1} \in S_{\alpha+1}) .$$

By the mapping

(2) $$u \to \bar{u} = \langle c_0, c_1, \ldots, c_\alpha, \ldots \rangle,$$

the group S_0 is isomorphically mapped into V. For any subgroup A of S_0, the image group under this mapping will be denoted by $\bar{A}$. In (2), the element u is contained in S_α if and only if $c_\beta = 0$ for every $\beta < \alpha$. Similarly, u is contained in P_α if and only if $c_\beta = 0$ for every $\beta \geqq \alpha$.

Let

$$\mathfrak{u} = \langle c_0, c_1, \ldots, c_\alpha, \ldots \rangle \quad (\alpha < \lambda)$$

be any vector in V. We denote by ${}_\mu[\mathfrak{u}]_\nu$ $(\mu < \nu \leqq \lambda)$ a vector

$$\langle c'_0, c'_1, \ldots, c'_\alpha, \ldots \rangle \quad (\alpha < \lambda)$$

such that $c'_\alpha = 0$ $(\alpha < \mu$ or $\alpha \geqq \nu)$ and $c'_\alpha = c_\alpha$ $(\mu \leqq \alpha < \nu)$. If a vector $\mathfrak{u}$ in V is contained in $\bar{S}_0$, then, for any μ, ν $(\mu < \nu \leqq \lambda)$, the vector ${}_\mu[\mathfrak{u}]_\nu$ is also contained in $\bar{S}_0$.

For any ordinal $\mu \leqq \lambda$, all the vectors (in V) of the form

$$\mathfrak{u} = \langle c_0, c_1, \ldots, c_\alpha, \ldots \rangle \quad (\alpha < \lambda),$$

where $c_\alpha = 0$ for every $\alpha \geqq \mu$, form a subgroup of V, which is denoted by V_μ.

Let ϱ $(0 < \varrho < \lambda)$ be any limit ordinal. We shall consider the following as fixed for ϱ.

Denote by $\bar{P}^*_\varrho$ the subgroup of V_ϱ, which consists of all vectors $\mathfrak{u}$ in V_ϱ such that ${}_0[\mathfrak{u}]_\alpha$ is contained in $\bar{P}_\varrho$ for any $\alpha < \varrho$. $\bar{P}_\varrho$ is included in $\bar{P}^*_\varrho$.

Let u be any element of $p^\varrho G$. For any ordinal $\alpha < \varrho$, take an element u_α in $p^\alpha G$ such that $pu_\alpha = u$. By the mapping (2), we have

(3) $$\begin{aligned} u_1 - u_0 &\to \langle c_{10}, c_{11}, c_{12}, \ldots, c_{1\alpha}, \ldots \rangle, \\ u_2 - u_0 &\to \langle c_{20}, e_{21}, c_{22}, \ldots, c_{2\alpha}, \ldots \rangle, \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ u_{\alpha+1} - u_0 &\to \langle c_{\alpha+1,0}, c_{\alpha+1,1}, c_{\alpha+1,2}, \ldots, c_{\alpha+1,\alpha}, \ldots \rangle \end{aligned}$$

Construct the new vector (in V_ϱ)

$$\mathfrak{u} = \langle c_{10}, c_{21}, \ldots, c_{\alpha+1,\alpha}, \ldots, \overset{\substack{\varrho\text{-th component} \\ \downarrow}}{0}, 0, \ldots \rangle \quad (\alpha < \varrho).$$

We call it a *diagonal vector* of u (relative to ϱ). In (3), we can prove

(4) $$c_{\alpha+1,\alpha} = c_{\gamma\alpha}$$

for any $\alpha < \varrho$ and for any γ $(\alpha < \gamma < \varrho)$. It follows from this equality, that any diagonal vector is contained in $\bar{P}^*_\varrho$. (We omit for a while the word 'relative to ϱ'.)

Let $\mathfrak{u}_1$, $\mathfrak{u}_2$ be diagonal vectors of u_1, u_2 respectively. Then, $\mathfrak{u}_1 + \mathfrak{u}_2$ is a diagonal vector of $u_1 + u_2$. When $\mathfrak{u}$ is a diagonal vector of u, a vector $\mathfrak{v}$ in $\bar{P}_\varrho$ is a diagonal vector of u if and only if $\mathfrak{v}$ is congruent to $\mathfrak{u}$ mod $\bar{P}^*_\varrho$.

Now we shall state the

Main theorem on diagonal vectors. *Let ϱ be any fixed limit ordinal such that $0 < \varrho < \lambda$. Let the coset of $\bar{P}^*_\varrho$ mod $\bar{P}_\varrho$, which consists of all the diagonal vectors of u, correspond to any element u of $p^\varrho G$. Then, we obtain a homomorphic mapping from $p^\varrho G$ into $\bar{P}^*_\varrho/\bar{P}_\varrho$, and the kernel of this mapping coincides with $p^{\varrho+1}G$.*

Proof. We have only to prove that the kernel coincides with $p^{\varrho+1}G$. Assume first that u is even contained in $p^{\varrho+1}G$. Then there exists an element v in $p^\varrho G$ such that $pv = u$. Put $u_\alpha = v$ for every $\alpha < \varrho$. The diagonal vector obtained from this system $\{u_\alpha\}$ is the zero vector, and is naturally contained in $\bar{P}_\varrho$. Conversely, assume that a diagonal vector of u is contained in $\bar{P}_\varrho$. Then the zero vector is also a diagonal vector of u. Thus we have the elements u_α such that

$$pu_\alpha = u, \quad u_\alpha \in p^\alpha G,$$

$$u_{\alpha+1} - u_0 \to \langle c_{\alpha+1,0}, \ldots, c_{\alpha+1,\alpha}, \ldots \rangle,$$

where $c_{\alpha+1,\alpha} = 0$ for every $\alpha < \varrho$. By the equality (4). we get

$$c_{\alpha+1,\beta} = c_{\beta+1,\beta} \quad (\beta \leqq \alpha),$$

and hence

$$c_{\alpha+1,\beta} = 0 \qquad (\beta \leqq \alpha).$$

This implies

$$u_{\alpha+1} - u_0 \to \langle 0, 0, \ldots, \underset{\substack{\uparrow \\ \alpha\text{-th component}}}{0}, c_{\alpha+1,\alpha+1}, \ldots \rangle,$$

and therefore

$$u_{\alpha+1} - u_0 \in S_{\alpha+1}.$$

Since $u_{\alpha+1} \in p^{\alpha+1}G$ and $S_{\alpha+1} = S_0 \cap p^{\alpha+1}G$, we obtain

$$u_0 \in p^{\alpha+1}G$$

for every $\alpha < \varrho$, which shows us

$$u_0 \in p^\varrho G.$$

Thus we have

$$u \in p^{\varrho+1}G,$$

which completes the proof.

Corollary 1. *Let $\{u_\gamma\}_{\gamma\in\Gamma}$ be a system of elements of $p^\varrho G$, and let $\mathfrak{u}_\gamma$ be a diagonal vector of u_γ, for every γ, respectively. If $\{u_\gamma\}_{\gamma\in\Gamma}$ is independent* mod $p^{\varrho+1}G$, *then $\{\mathfrak{u}_\gamma\}_{\gamma\in\Gamma}$ is independent* mod $\bar{P}_\varrho$.

For the ranks of groups we have

COROLLARY 2. $r(\bar{P}_\varrho^*/\bar{P}_\varrho) \geqq r(p^\varrho G/p^{\varrho+1}G)$.

Let μ be an ordinal $< \lambda$. To any vector $\mathfrak{u}$ of V, we let correspond the vector ${}_\mu[\mathfrak{u}]_\lambda$. It gives an endomorphism of V, and is denoted by φ_μ. For any $\mu < \varrho$

$$\varphi_\mu \bar{P}_\varrho^*/\varphi_\mu \bar{P}_\varrho \cong \bar{P}_\varrho^*/\bar{P}_\varrho$$

holds. The α-th *Ulm invariant* of G, i.e.

$$r(S_\alpha/S_{\alpha+1}) = r(C_\alpha),$$

is denoted by $\mathfrak{a}_\alpha$.

COROLLARY 3.

$$\text{Min}\, r\,(\varphi_\mu \bar{P}_\varrho^*) \geqq \sum_{n<\omega} \mathfrak{a}_{\varrho+n}\,. \tag{5}$$

(ω denotes the least infinite ordinal. We consider, of course, only $\varrho + n$'s such that $\varrho + n < \lambda$.)

We shall apply our Main Theorem to the case of principal p-groups.

When, for a suitable choice of $\{P_\alpha\}$,

$$S_0 = \sum_{\alpha<\lambda} C_\alpha\,,$$

G is called a *principal p-group*. A reduced Abelian p-group G of length λ is a principal p-group if and only if the socle S_0 of G allows a basis $\mathfrak{c}$ such that, for any $\alpha < \lambda$, $\mathfrak{c} \cap p^\alpha G$ is a basis of $S_\alpha = S_0 \cap p^\alpha G$.

REMARK. We may call such a basis $\mathfrak{c}$ a *principal system* of G, though A. Kertész [3] defined by this a weaker concept.

A subgroup H of a reduced Abelian p-group G is said to be *height-finite* when the set of the heights (in G) of all the elements of H is a finite one. If the socle of G is the union of a countable ascending sequence of height-finite subgroups of G, then G is a principal p-group. It follows that any (at most) countable reduced Abelian p-group and also any direct sum of such p-groups are principal p-groups. By the result in the author's paper [2], any forest p-group is a principal p-group.

When G is a principal p-group, the structure of $\varphi_\mu \bar{P}_\varrho^*$ is uniquely determined by μ, ϱ and the Ulm invariants $\{\mathfrak{a}_\alpha\}$ $(\alpha < \varrho)$. We denote by $\mathfrak{p}(\mathfrak{a}_\alpha|\alpha < \varrho)$ the cardinal appeared on the left side of (5), i.e.

$$\mathfrak{p}\,(\mathfrak{a}_u|\alpha < \varrho) = \text{Min}\, r\,(\varphi_\mu \bar{P}_\varrho^*)\,.$$

THEOREM. *Let G be any principal p-group of length λ and let the α-th Ulm invariant of G be $\mathfrak{a}_\alpha$ for every $\alpha < \lambda$. Then we have*

1) *$\lambda \leqq \Omega$ (the least non-countable ordinal),*

2) *for any limit ordinal ϱ $(0 < \varrho < \lambda)$, it holds that*

$$\mathfrak{p}\,(\mathfrak{a}_\alpha|\alpha < \varrho) \geqq \sum_{n<\omega} \mathfrak{u}_{\varrho+n}\,.$$

PROOF. Since 2° is included in Corollary 3 to the Main Theorem, we have only to prove property 1°. Take any limit ordinal ϱ $(0 < \varrho < \lambda)$. Since G is a reduced p-group of length λ, we have

$$p^{\varrho} G \neq p^{\varrho+1} G.$$

Consider an element u of G such that $u \in p^{\varrho}G$, $u \notin p^{\varrho+1}G$. Let

$$\mathfrak{u} = \langle c_0, c_1, \ldots, c_\alpha, \ldots \rangle$$

be a diagonal vector of u (relative to ϱ). By the Main Theorem, it holds that

$$\mathfrak{u} \in \bar{P}_\varrho^*, \ \mathfrak{u} \notin \bar{P}_\varrho.$$

The former implies that, for any $\mu < \varrho$, c_α $(\alpha < \mu)$ is equal to 0 except for a finite number of α. On the other hand, the latter implies that, for any $\nu < \varrho$, there exists α $(\nu \leqq \alpha < \varrho)$ such that $c_\alpha \neq 0$. Therefore, when we collect all the ordinals α with $c_\alpha \neq 0$, we obtain a countable ascending sequence of ordinals

$$\alpha_0 < \alpha_1 < \alpha_2 < \ldots \quad (< \varrho)$$

having ϱ as its limit, i.e., for any $\nu < \varrho$, there exists k such that $\nu \leqq \alpha_k$. Such a situation holds for any limit ordinal ϱ $(0 < \varrho < \lambda)$, from which we can conclude that $\lambda \leqq \Omega$. In fact, assume that $\lambda > \Omega$. Then we can put $\Omega = \varrho$, and we get a countable ascending sequence

$$\alpha_0 < \alpha_1 < \alpha_2 < \ldots \quad (< \Omega)$$

having Ω as its limit. Since every α_i $(< \Omega)$ is an at most countable ordinal, it follows that Ω is a countable ordinal, which is a contradiction, q.e.d.

Conversely, it holds the following

EXISTENCE THEOREM FOR PRINCIPAL p-GROUPS. *Let* $\{\mathfrak{a}_\alpha\}$ $(\alpha < \lambda)$ *be a system of (not necessarily distinct) cardinals, where the index* α *ranges over the set of all ordinals smaller than the given ordinal* λ. *We assume that, for any* $\gamma < \lambda$, *there exists a non-zero* $\mathfrak{a}_\mu$ *such that* $\gamma \leqq \mu < \lambda$. *Then, there exists a principal p-group G of length* λ, *whose* α*-th Ulm invariant is equal to* $\mathfrak{a}_\alpha$ *for every* $\alpha < \lambda$, *if the following conditions are satisfied:*

1) $\lambda \leqq \Omega$.

2) *For any limit ordinal* ϱ $(0 < \varrho < \lambda)$, *it holds that*

$$\mathfrak{p}\,(\mathfrak{a}_\alpha | \alpha < \varrho) \geqq \sum_{n < \omega} \mathfrak{a}_{\varrho+n} \, .$$

The proof of this theorem proceeds similarly to that of the Existence Theorem for forest p-groups [2], but it is much more complicated, and the author intends to publish it elsewhere.

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Addendum in the proof-reading. Cf. the author's recent paper, Realism in the theory of Abelian groups, III, *Comm. Math. Univ. St. Paul*, **12** (1964), 75—111. It contains the complete proofs of the above assertions.

ON GENERATING SUBGROUPS OF ABELIAN GROUPS[1]

By

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The present paper resulted from a discussion at the Symposium on Abelian Groups held at New Mexico State University [4].

In his paper on torsion-free groups of infinite rank J. D. Reid [5] has given a necessary and sufficient condition for a torsion-free group to be generated by two of its free subgroups. This condition provided an answer in the torsion-free case to a question of Khabbaz, *What Abelian groups are generated by two of their subgroups each of which is a direct sum of cyclic groups.* This paper answers Khabbaz's question in the general case.

To facilitate the discussion we call a group which is generated by n subgroups, each of which is a direct sum of cyclic groups, $\sum^n$-cyclic. Thus, in particular, a direct sum of cyclic groups is $\sum^1$-cyclic, the groups generated by two direct sums of cyclic groups are $\sum^2$-cyclic, and so on.

We begin with a discussion of $\sum^2$-cyclic primary groups and prove that every reduced primary group is $\sum^2$-cyclic. A characterization of $\sum^2$-cyclic primary groups is that they are those p-groups which are not a finite direct sum of $Z(p^\infty)$'s directly summed with a group of bounded order. Then we give a necessary and sufficient condition that a torsion group be Σ^2-cyclic. With the help of Reid's result, we then pass to the mixed case and give the following characterization of these groups: A group G is $\sum^2$-cyclic if and only if (a) G/T is free of finite rank and T is $\sum^2$-cyclic or (b) the torsion-free rank of G is infinite. A rather surprising result in this direction is: an Abelian group G is $\sum^n$-cyclic for some n if and only if G is $\sum^2$-cyclic. In the second part of the paper, we characterize the p-groups that are generated by two basic subgroups and show that a p-group G is generated by n of its basic subgroups for some integer n if and only if G is generated by two of its basic subgroups. Finally we discuss the most general class of $\sum^2$-cyclic groups that are closed under subgroups and pose a few interesting questions in this direction.

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And now a word on notation:

1. Let $h(g)$ denote the p-height of g in G.
2. $|G|$ means the cardinal of G.
3. $o(g)$ means the order of the element g.
4. $\langle g\rangle$ denotes the cyclic group generated by the element g.
5. T and G_t will both be used to denote the maximal torsion subgroup of G.
6. $r(G)$ denotes the rank of G.
7. Z stands for the additive group of the integers.
8. $+$ means sum not necessarily direct.
9. $\oplus$ means direct sum.
10. $G[p^m] = \{x \in G : p^m x = 0\}$.
11. $B = \sum_{i=1}^{\infty} C(p^i)$ is referred to as a standard B.
12. $\bar{B}$ is the closure of B (see closed groups in Fuchs [1]).
13. The word 'plus' is used to mean $\oplus$.

For the most part, the notation is that of L. Fuchs [1]. We refer informally to [1] as Fuchs and [2] as Kaplansky.

1. On $\sum^2$-cyclic groups

THEOREM 1. *Let G be a p-group containing a subgroup $K = \sum_{\alpha\in A}\langle x_\alpha\rangle$ with $|p^n K| = |G|$ for all n (i.e. $|\{x_\alpha : o(x_\alpha) \geqq p^n\}| = |G|$ for all n). Then G is $\sum^2$-cyclic, $G = \{H, K\}$ where $H \cong K$.*

PROOF. Obviously it suffices to prove the theorem for $G \neq 0$. To this end, notice that A is infinite, $|A| = |G|$, and let $G = \{g_\beta : \beta \in B\}$ where B is initially well ordered. For each $\beta \in B$ define an element $a_\beta \in A$ as follows: If 1 is the first element of B, let a_1 be an element of A such that

1) $o(x_{a_1}) \geqq o(g_1)$,

2) x_{a_1} does not occur in any expression of any element of $\langle g_1\rangle \cap K$, i.e., if $g \in \langle g_1\rangle \cap K$ and $g = \sum_{i=1}^{n} n_i x_{a_i} \neq 0$, n_i integers, $a_i \in A$ and $a_1 \neq$ $\neq a_i$, $2 \leqq i \leqq n$, then $n_1 x_{a_1} = 0$.

This defines a_1. Next suppose that a_β is defined for all $\beta < \beta_0$. Define a_{β_0} as follows: let K_{β_0} be the subgroup of G generated by $\{g_\beta : \beta \leqq \beta_0\}$ and $\{x_{a_\beta} : \beta < \beta_0\}$. Let a_{β_0} be an element of A satisfying:

1) $o(x_{a_{\beta_0}}) \geqq o(g_{\beta_0})$,

2) $x_{a_{\beta_0}}$ does not occur in an expression of any element of $K_{\beta_0} \cap K$ as an element of K. Note that $|K_{\beta_0} \cap K| < |G|$.

Observe next that a_β has been defined by transfinite induction for all $\beta \in B$. To see that the a_β are all distinct, suppose that for $\beta_1 < \beta_2$ we have

$\alpha_{\beta_1} = \alpha_{\beta_2}$. Then $x_{\alpha_{\beta_2}} = x_{\alpha_{\beta_1}} \in K_{\beta_2} \cap K$, contrary to the definition of $x_{\alpha_{\beta_2}}$. Thus $|\{\alpha_\beta : \beta \in B\}| = |B|$.

Now for any $\alpha \in A$ let y_α be an element of G defined as follows:

(a) If $\alpha = \alpha_\beta$ for some $\beta \in B$, let $y_\alpha = g_\beta$.

(b) If $\alpha \neq \alpha_\beta$ for all $\beta \in B$, let $y_\alpha = 0$.

Evidently for all $\alpha \in A$, $o(x_\alpha + y_\alpha) = o(x_\alpha)$, for otherwise $0 \neq p^n x_{\alpha_\beta} = -p^n g_\beta \in K_\beta \cap K$ is a contradiction. Moreover, $\{x_\alpha + y_\alpha : \alpha \in A\}$ is an independent set. To see this, let $\sum_{i=1}^{n} n_i(x_{\alpha_i} + y_{\alpha_i}) = 0$ where $y_{\alpha_j} = g_{\beta_j}$ is the non-zero y_{α_i} with the largest subscript occurring such that $n_i(x_{\alpha_i} + y_{\alpha_i}) \neq 0$ (if all $y_{\alpha_i} = 0$, it follows from the independence of the set $\{x_\alpha : \alpha \in A\}$ that the terms are zero). Then $\sum n_i\, x_{\alpha_i} = -\sum n_i\, y_{\alpha_i} \in K_{\beta_j} \cap K$ and in particular $0 \neq n_j\, x_{\alpha_{\beta_j}}$ occurs in an expression of an element of $K_{\beta_j} \cap K$ as an element of K.

From the independence of $\{x_\alpha + y_\alpha\}_{\alpha \in A}$ we have that $H = \sum_{\alpha \in A} \langle x_\alpha + y_\alpha \rangle$ is indeed a direct sum of cyclic groups. That $H \simeq K$ follows trivially from the fact that $o(x_\alpha + y_\alpha) = o(x_\alpha)$ for all $\alpha \in A$. Finally, that $G = \{H, K\}$ follows from the fact that each g_β occurs as a y_α in $x_\alpha + y_\alpha$ for some $\alpha \in A$, and Theorem 1 is proved.

Remark. Notice that with a few slight modifications such as deleting the requirement on order, the proof of Theorem 1 can be used to prove Reid's result that every torsion-free group of infinite rank is $\sum^2$-cyclic.

We now proceed with a few lemmas; with the help of Theorem 1 they lead to a proof that every reduced torsion group is $\sum^2$-cyclic.

Lemma 1. *Let H be a pure subgroup of a group G where $\{x_\alpha + H\}_{\alpha \in A}$ is an independent set in G/H with $o(x_\alpha) = o(x_\alpha + H)$. Let L be the subgroup generated by $\{x_\alpha\}_{\alpha \in A}$. Then $L \cap H = 0$ and $L = \sum_{\alpha \in A} \langle x_\alpha \rangle$.*

Proof. The lemma follows immediately from the proof of Theorem 5 of Kaplansky.

Lemma 2. *Let G be a p-group, with $|G^1[p]| = |G|$. Then G is $\sum^2$-cyclic. ($G^1 = \bigcap_n p^n G$.)*

Proof. It suffices to prove the lemma for $G \neq 0$. In this case, $|G| = |G^1[p]|$ is infinite. Now partition $G^1[p]$ into summands, $\{N_i\}_{i=1}^{\infty}$ with $|N_i| = |G|$. Clearly, for each $i \geqq 1$, there exists a subgroup $C_i = \sum_\lambda C_\lambda(p^i)$ such that $C_i[p] = N_i$. The subgroup $K = \sum_{i=1}^{\infty} C_i$ satisfies the hypotheses of Theorem 1, so that G is $\sum^2$-cyclic.

Corollary. *A divisible p-group of infinite rank is $\sum^2$-cyclic.*

The first of our main results is

Theorem 2. *Every reduced primary group is $\sum^2$-cyclic.*

Proof. Let G be a reduced p-group. If G is $\sum^1$-cyclic, G is trivially

$\sum^2$-cyclic. If G is not $\sum^1$-cyclic, $p^nG \neq 0$ for any n, whence G has infinite final rank (Fuchs, p. 105). Thus there exists B basic in G with G/B of infinite rank. Now G/B as a divisible group of infinite rank satisfies the hypotheses of Lemma 2. Thus G/B is $\sum^2$-cyclic with $G/B = C_1/B + C_2/B$ where C_1/B and C_2/B are $\sum^1$-cyclic. Since B is pure in G it follows easily from Theorem 5 of Kaplansky that C_1 and C_2 are $\sum^1$-cyclic. It is clear that $C_1 + C_2 = G$ and the proof is complete.

Remark. The generating subgroups C_1, C_2 in the foregoing proof may be chosen to be isomorphic. To see this use Theorem 1 to obtain $C_1/B \cong C_2/B$, from whence $C_1 \cong C_2$ follows easily.

A corollary to the proof of the foregoing is

Theorem 3. *Let G be a p-group with fin $r(G) = r(G) \geqq \aleph_0$. Then G is $\sum^2$-cyclic.*

Now we give a necessary and sufficient condition that a primary group be $\sum^2$-cyclic. This condition together with Theorems 5 and 6 will characterize $\sum^2$-cyclic torsion groups.

Theorem 4. *A p-group G is $\sum^2$-cyclic if and only if*

(a) *G is reduced* or

(b) *The divisible part of G has infinite rank* or

(c) *The reduced part of G is unbounded.*

Proof. If condition (a) is satisfied, Theorem 2 gives us that G is $\sum^2$-cyclic. Now write $G = D \oplus R$ (i.e., as divisible $\oplus$ reduced), and when D has infinite rank, D is $\sum^2$-cyclic. Thus both D and R are $\sum^2$-cyclic, and clearly this yields that G is $\sum^2$-cyclic. If condition (c) is satisfied and neither (a) nor (b) is satisfied we have for the subgroup R in $G = D \oplus R$ that the rank of D is finite and $R = H \oplus K$ where H is bounded and the final rank of K is equal to the rank of K (Fuchs, p. 106). Hence the final rank of $L = D \oplus K$ is equal to the rank of L. Thus, by Theorem 3, L is $\sum^2$-cyclic, and now it is clear that G is $\sum^2$-cyclic.

To prove the converse, now suppose that none of the conditions (a), (b), or (c) is satisfied. Then expressing $G = D \oplus R$ as before, we have that $0 \neq D$ is of finite rank and R is of bounded order. Let the order bound of R be p^n and G be $\sum^2$-cyclic. Then $p^nG = D$ would be $\sum^2$-cyclic. But this is impossible because the fact that D is of finite rank gives us that D is finitely generated and, hence, a direct sum of cyclic groups. This contradiction completes the proof of Theorem 9.

As a corollary to Theorem 4 we obtain easily

Theorem 5. *A p-group is $\sum^2$-cyclic if and only if G is not a direct sum of (at least one) finitely many $Z(p^\infty)$ groups and a group of bounded order.*

Passing next from p-groups to torsion groups we have

Theorem 6. *A torsion group T is $\sum^2$-cyclic if and only if each of its primary components T_p is $\sum^2$-cyclic.*

PROOF. To prove that if T_p is $\sum^2$-cyclic for each p, T is $\sum^2$-cyclic is straightforward. The converse follows easily from the fact that any subgroup of a direct sum of cyclic groups is a direct sum of cyclic groups.

COROLLARY. *Every reduced torsion group is $\sum^2$-cyclic.*

PROOF. Theorems 2 and 6.

The $\sum^2$-cyclic torsion groups having been characterized, we now state Reid's result for the torsion-free case and pass to a discussion of mixed $\sum^2$-cyclic groups.

THEOREM 7. (Reid). *Let G be a torsion-free group. Then G is $\sum^2$-cyclic if and only if G is free or has infinite rank.*

In the discussion of mixed groups G we write T for the maximal torsion subgroup of G.

THEOREM 8. *A mixed group G is $\sum^2$-cyclic if both T and G/T are $\sum^2$-cyclic.*

PROOF. Suppose that T and G/T are $\sum^2$-cyclic. That G is $\sum^2$-cyclic follows easily from the definition of $\sum^2$-cyclic and Theorem 5 of Kaplansky.

A partial converse of Theorem 8 is

THEOREM 9. *Let G be $\sum^2$-cyclic. Then G/T is $\sum^2$-cyclic.*

PROOF. Let $G = M + N$ where M and N are $\sum^1$-cyclic. Express $M = F_1 \oplus T_1$, $N = F_2 \oplus T_2$ where F_i is free and T_i is torsion $\sum^1$-cyclic for $i = 1, 2$. Then $G/T = \dfrac{F_1 \oplus T}{T} + \dfrac{F_2 \oplus T}{T}$, and G/T is $\sum^2$-cyclic since $F_i \simeq \dfrac{F_i \oplus T}{T}$, $i = 1, 2$.

REMARK. If G is $\sum^2$-cyclic, T does not have to be $\sum^2$-cyclic as the following example shows. Let $F = \sum_{n=1}^{\infty} \langle 1_n \rangle$ where $\langle 1_n \rangle \simeq Z$, and $Z(p^\infty) = \{x_1, x_2, \ldots\}$. Then $G = F \oplus Z(p^\infty) = \sum_n \langle 1_n \rangle + \sum_n \langle 1_n + x_n \rangle$ is $\sum^2$-cyclic, but $T = Z(p^\infty)$ is not. Thus the converse of Theorem 8 does not hold.

THEOREM 10. *Let G be a mixed group such that $r(G/T) \geqq \aleph_0$. Then G is $\sum^2$-cyclic.*

PROOF. By Theorems 7 and 8 it suffices to prove that $r(G/T) \geqq \aleph_0$ and T not $\sum^2$-cyclic imply that G is $\sum^2$-cyclic. To this end suppose that $r(G/T) \geqq \aleph_0$ and T is not $\sum^2$-cyclic. By Theorems 5 and 6, T has at least one primary component T_p which is expressible as $T_p = D_p \oplus B$ where $0 \neq D_p$ is a direct sum of finitely many $Z(p^\infty)$ and B is of bounded order. Set $D = \sum_p D_p$ ($D_p = 0$ whenever the divisible part of T_p has infinite rank) and notice that $|D| = \aleph_0$. Write out the elements of D as $\{d_1, d_2, d_3, \ldots\}$. Put $T_p = D_p \oplus T'_p$ for each p. Then we have $T = D \oplus T_0$ where $T_0 = \sum_p T'_p$. But by Theorems 4 and 6, T_0 is $\sum^2$-cyclic so that $T_0 = T_1 + T_2$ where T_1 and T_2 are $\sum^1$-cyclic. Since $r(G/T) \geqq \aleph_0$, we have that G/T is

$\sum^2$-cyclic by Theorem 7. We wish, however, to construct the two generating free subgroups of G/T as in the proof of Theorem 1 (see also Remark following Theorem 1). To this end let $\sum_{\alpha\in A} \langle x_\alpha\rangle$ in the proof of Theorem 1 be replaced in G/T by $\sum_{n=1}^{\infty} \langle x_{\alpha_n} + T\rangle \oplus \sum_{\beta} \langle x_\beta + T\rangle$ where the β's come from an index set with the same cardinal as $r(G/T) = |\, G/T\,|$. Now replace $\sum \langle x_\alpha + {} + y_\alpha\rangle$ in the proof of Theorem 1 by $\sum_{n=1}^{\infty} \langle x_{\alpha_n} + d_n + T\rangle \oplus \sum_{\beta} \langle x_\beta + y_\beta + T\rangle$ where the latter group is obtained from $\sum_{\beta} \langle x_\beta + T\rangle$ as in the proof of Theorem 1. Let $L_1 = \sum_{n} \langle x_{\alpha_n}\rangle \oplus \sum_{\beta} \langle x_\beta\rangle$, $L_2 = \sum_{n} \langle x_{\alpha_n} + d_n\rangle \oplus \sum_{\beta} \langle x_\beta + y_\beta\rangle$. By the construction in the proof of Theorem 1 we have that

$$G/T = \Big(\sum_{n} \langle x_{\alpha_n} + T\rangle \oplus \sum_{\beta} \langle x_\beta + T\rangle\Big) + {} + \Big(\sum_{n} \langle x_{\alpha_n} + d_n + T\rangle \oplus \sum_{\beta} \langle x_\beta + y_\beta + T\rangle\Big)$$

so that clearly $G/T = \dfrac{L_1 \oplus T}{T} + \dfrac{L_2 \oplus T}{T}$. Obviously, $D \subset L_1 + L_2$ and from this it follows easily that $G = L_1 \oplus T_1 + L_2 \oplus T_2$ and G is $\sum^2$-cyclic. This concludes the proof of Theorem 10.

Combining a few of the preceding results we obtain the following characterization of $\sum^2$-cyclic Abelian groups.

Theorem 11. *A group G is $\sum^2$-cyclic if and only if G satisfies one of the following conditions:*

(a) *G/T is free of finite rank and T is $\sum^2$-cyclic.*

(b) $r(G/T) \geqq \aleph_0$.

Proof. Suppose condition (a) holds. Then $G/T = \dfrac{F \oplus T}{T}$ where F is free. Since $T = T_1 + T_2$ with T_1 and T_2 $\sum^1$-cyclic, $G = (F \oplus T_1) + T_2$, and G is $\sum^2$-cyclic. If condition (b) holds, G is $\sum^2$-cyclic by Theorem 10.

To prove the converse suppose that G is $\sum^2$-cyclic. Then by Theorem 9, G/T is $\sum^2$-cyclic. If $r(G/T)$ is finite, then G/T is finitely generated (since G/T is $\sum^2$-cyclic) and hence free of finite rank. To complete the proof it suffices to verify that T is $\sum^2$-cyclic. Now write $G = M + N$ where M and N are $\sum^1$-cyclic and put $M = F_1 \oplus T_1$ and $N = F_2 \oplus T_2$, where F_1 and F_2 are free of finite rank and T_1 and T_2 are torsion $\sum^2$-cyclic. Since $F_1 + F_2$ is finitely generated, we have that $F_1 + F_2$ is $\sum^1$-cyclic. Thus $F_1 + F_2 = {} = F \oplus T_3$ where F is free and T_3 is torsion $\sum^1$-cyclic and finite. Hence $T_1 + T_3$ is $\sum^1$-cyclic and clearly $T = (T_1 + T_3) + T_2$. To see that $T_1 + T_3$ is $\sum^1$-cyclic first observe that for some integer n, $n(T_1 + T_3) \subset T_1 \subset$

$\subset T_1 + T_3$ since T_3 is finite. Thus since T_1 is $\sum^1$-cyclic, $T_1 + T_3$ is $\sum^1$-cyclic by Theorem 12.4 in Fuchs. Whence T is $\sum^2$-cyclic and the proof is complete.

THEOREM 12. *Let G be $\sum^2$-cyclic. Then nG is $\sum^2$-cyclic for all n. Moreover, suppose nG is $\sum^2$-cyclic for some positive integer n. Then G is $\sum^2$-cyclic.*

PROOF. If G is $\sum^2$-cyclic with $G = H + K$, then $nG = nH + nK$. For the second part use Theorems 5, 6, and 11.

THEOREM 13. *A group G is $\sum^n$-cyclic for some n if and only if G is $\sum^2$-cyclic.*

PROOF. If G is $\sum^2$-cyclic put $n = 2$. Now to prove the converse we see that by Theorem 11 it suffices to show that if $r(G/T) < \infty$ then G/T is free of finite rank and T is $\sum^2$-cyclic. To this end let G be $\sum^n$-cyclic and $r(G/T) < \infty$. Put $G = \{H_1, \ldots, H_n\}$ where each H_i is $\sum^1$-cyclic and $H_i = F_i \oplus T_i$, $1 \leq i \leq n$, with F_i free and T_i $\sum^1$-cyclic. Clearly $r(G/T) < \infty$ yields that the F_i are of finite rank so that as a finitely generated group G/T is free of finite rank. It remains to show that T is $\sum^2$-cyclic. For this purpose we first show that T is $\sum^n$-cyclic. To see this, notice that $H = \{F_1, F_2, \ldots, F_n\}$ is finitely generated whence H_t is also finitely generated. Thus $H_t + T_i$ is $\sum^1$-cyclic since for $m = o(H_t)$ we have that $m(H_t + T_i) \subset T_i$. Clearly, $T = \{H_t + T_1, H_t + T_2, \ldots, H_t + T_n\}$ and T is $\sum^n$-cyclic as stated. Clearly T $\sum^n$-cyclic yields T_p $\sum^n$-cyclic for each primary component T_p. So by Theorem 6 we are left with showing that T $\sum^n$-cyclic implies T $\sum^2$-cyclic in the primary case. Now suppose that T is a p-group which is not $\sum^2$-cyclic; then by Theorem 5, $T = D \oplus B$ where $0 \neq D$ is a finite direct sum of $Z(p^\infty)$ and B has order bound p^r. Let $T_1, T_2, \ldots, T_n$ be $\sum^1$-cyclic and generate T. Notice that $T_i + B$ is $\sum^1$-cyclic since $p^r(T_i + B) = p^r T_i$. Since B is a direct summand of $T_i + B$, $(T_i + B)/B$ is $\sum^1$-cyclic and as a subgroup of $T/B \cong D$ must be finitely generated. But the groups $(T_i + B)/B$, $1 \leq i \leq n$, generate T/B so that T/B is finitely generated, and hence $\sum^1$-cyclic. This contradicts the divisibility of $D \cong T/B$ and Theorem 13 is proved.

2. On groups generated by two basic subgroups

In this section we concern ourselves with the question of what p-groups are generated by two basic subgroups.

To this end we make the following definition:

DEFINITION 1. A primary group is called *starred* [3] if and only if $|G| = |B|$ where B is basic in G.

The next definition concerns the decomposition in a theorem of Baer [1, p. 98] which is as follows: assume that B is a subgroup of p-group G and $B = \sum B_n$ where B_n is the direct sum of cyclic groups of order p^n.

Then B is a basic subgroup of G if and only if

$$G = B_1 \oplus \ldots \oplus B_n \oplus \{B_n^*, p^n G\}$$

for every n where

$$B_n^* = B_{n+1} \oplus B_{n+2} \oplus \ldots$$

Let $B = \sum_n B_n$ be basic in a p-group G. Put $S_n = \sum_{k=1}^{n} B_k$, $B_n^* = \sum_{k>n} B_k$, and $G_n = \{B_n^*, p^n G\}$, and B_n^* is basic in G_n. Then for each n, $G = S_n \oplus G_n$.

Definition 2. With Baer's Theorem in mind, we call a p-group G *strongly starred* if each G_n is starred.

The following lemma is useful in obtaining our characterization of p-groups generated by two basic subgroups.

Lemma 3. *Let $B = \sum_k B_k$ be basic in a starred p-group G such that for each $n \geq 1$ there exists $n_k > n$ with $|B_{n_k}| > |B_n|$. Then $G = K \oplus L$ where $K = \sum K_{n_k}$ with $|K_{n_k}| = |B_{n_k}|$ and $K_{n_k} = \sum C(p^{n_k})$ for all $k \geq 1$. Moreover $|K| = |G|$.*

Proof. See [3, p. 529, cases 3 and 4].

The next two lemmas will be used in the proof of the main theorem of this section.

Lemma 4. *Let $G = H \oplus K$ be a p-group with f a homomorphism from H onto K. Let $L = \{x + f(x) : x \in H\}$. Then $G = L \oplus K$.*

Proof. First we show that $L \cap K = 0$. For this purpose let $g \in L \cap K$. Then $g = x + f(x) = f(y)$ where $x, y \in H$. Thus $x = f(y - x) \in H \cap K$. Hence $g = 0$ and $L \cap K = 0$. To see that $G = L \oplus K$, let $z \in G$. Then $z = h + k = h + f(h) + (k - f(h))$ where $h \in H$ and $k \in K$. But $h + f(h) \in L$ and $k - f(h) \in K$ so that $z \in L \oplus K$.

Lemma 5. *Let $G = H \oplus K$ be a p-group, $H = \sum_{i=1}^{\infty} \langle x_i \rangle$ with f a homomorphism from H onto K such that $o(x) > o(f(x))$ for every $o \neq x \in H$. Let $L = \{x + f(x) : x \in H\}$. Then $L = \sum_{i=1}^{\infty} \langle x_i + f(x_i) \rangle$.*

Proof. To see this observe that if $\sum a_i(x_{k_i} + f(x_{k_i})) = 0$ we have $\sum a_i x_{k_i} = -\sum a_i f(x_{k_i})$ and thus $\sum a_i x_{k_i} = 0$ which implies that $a_i = 0$ for every i.

The main theorem in this section is

Theorem 14. *A p-group G is generated by two of its basic subgroups if and only if G is strongly starred.*

Proof. Let G be strongly starred with B basic in G. Write $G = S_n \oplus G_n$ with $r(p^n G) = \text{fin } r(G)$. We may assume that G is infinite and unbounded. Then we have the following cases.

Case I: Infinitely many B_j satisfy $|B_j| = |G_n|$.

In this case define a homomorphism f from B into G as follows:

1) If $|B_n| < |G|$ put $f = 0$ on B_n.

2) If $|B_n| = |G|$ maps B_n onto $(p^nG)[p^n]$ so that the generators of B_n do not all go onto 0.

Extending by linearity we obtain a homomorphism f of B into G which satisfies $h(x) < h(f(x))$ whenever $f(x) \neq 0$ ($h(g)$ denotes the height of g in G). To see this write $x = b_1 + b_2 + b_3 + \dots + b_n$ where $b_i \in B_i$. Now either $f(b_i) = 0$ or $h(b_i) < h(f(b_i))$. This together with $h(x) = \min_{1 \leq i \leq n} h(b_i)$ gives us $h(x) < h(f(x))$ whenever $f(x) \neq 0$. Hence for all $x \in B$ we have $h(x + f(x)) = h(x)$. Now set $S_n = \sum_{i=1}^{n} B_i$, $S'_n = \{x + f(x) : x \in S_n\}$, and let S denote the subgroup of G generated by the S'_n for all n.

Next we show that S is basic in G and $G = B + S$. To this end we first show that S is basic in G. To see that S is pure in G, suppose that $p^ng = s \in S$. Then $s = x + f(x)$ with $x \in S_m$ for some m. Since $h(x) = h(x + f(x)) \geqq n$, there exists $b \in B$ with $p^nb = x$. Hence $s = p^nb + f(p^nb) = p^n(b + f(b))$ where $b + f(b) \in S$ and S is pure in G. To see that S is a direct sum of cyclic groups, observe that if $\sum_{i=1}^{r} a_i(b_i + f(b_i)) = 0$, we have $\sum a_i b_i = -f(\sum a_i b_i)$. Now if $f(\sum a_i b_i) = 0$ we have $a_i \equiv 0 \bmod o(b_i)$ from the independence of the b_i. If $f(\sum a_i b_i) \neq 0$ we obtain a contradiction to $h(x) < h(f(x))$ when $f(x) \neq 0$. To show that G/S is divisible, it suffices to verify that $G = S + pG$. Since B is basic in G we may write $G = B + pG$. Let $g \in G$ and write $g = b + pz$, $b \in B$ and $z \in G$. But $g = (b + f(b)) + (pz - f(b))$ and $b + f(b) \in S$ and $pz - f(b) \in pG$ since $f(b) = 0$ or $h(f(b)) > h(b)$. Thus G/S is divisible and S is basic in G as stated.

To see that $G = B + S$, let $g \in G$ with $o(g) = p^r$. Then let $n > r$ be chosen so that $|B_n| = |G|$. By the purity of B and the divisibility of G/B, we may write $g = b + p^n z$ where $o(p^n z) \leqq o(g) = p^r$, $b \in B$, $z \in G$. Thus $p^n z \in (p^n G)[p^n]$. By the construction of the homomorphism f, there exists $x \in S_n \subset B$ with $f(x) = p^n z$. Now $g = b + p^n z = (b - x) + (x + f(x))$, and since $b - x \in B$ and $x + f(x) \in S$, we have $G = B + S$ as stated.

Case II: Only finitely many B_j satisfy $|B_j| = |G_n|$ and $|G_n| > \aleph_0$ for all n. Without loss of generality we may suppose that n is large enough so that for all $j > n$ we have $|B_j| < |G_n|$ and $|B_n^*| = \sum_{k \geq n+1} |B_k| = |G_n|$. Since G is strongly starred, we have by Lemma 3 that $G_n = K \oplus L$ where $K = \sum_k K_{n_k}$ is a direct sum of cyclic groups with $|K_{n_k}| < |K_{n_{k+1}}|$ for all $k \geqq 1$ and $\sum_k |K_{n_k}| = |G_n|$. By partitioning the generators of each K_{n_k} onto $\aleph_0$ disjoint subsets each having the same cardinality as K_{n_k} we easily obtain a subgroup $C = \sum_{j=1}^{\infty} C_j$ of K with $|C_j| = |G_n|$ for all j ($C_j =$

$= \sum_k (\sum C(p^{n_k})))$, and a natural homomorphism of K onto C. This homomorphism gives rise to a homomorphism f_1 from K onto L, and hence there exists a homomorphism f mapping $S_n \oplus K$ onto L (set $f = 0$ on S_n). We may assume this to be chosen so that $o(x) > o(f(x))$. Now $G = (S_n \oplus K) \oplus L$. Put $N = \{x + f(x) : x \in S_n \oplus K\}$. Then if B_L is basic in L, we have by Lemmas 4 and 5 that $B' = N \oplus B_L$ is basic in G. Extending K to a basic subgroup B of G we have $G = B + B'$. This concludes the proof of Case II.

CASE III: Only finitely many B_j satisfy $|B_j| = |G_n|$ and $|G_n| = \aleph_0$ for some n.

Let $G = D \oplus R$ where $D \neq 0$ is divisible and R is reduced. Let B be a basic subgroup of R. Then B is a basic subgroup of G, hence $|B| = |G|$ since G is starred. Now R is unbounded. To see this suppose that $p^n R = 0$ for some integer n. Then $p^n G = p^n D \oplus p^n R = D = G_n$ which contradicts the hypothesis that G is strongly starred. Thus, by problem 19a, p. 143, in Fuchs, we have that $R = K \oplus L$ where K is a direct sum of cyclic groups of unbounded order. Partition K into $\aleph_0$ disjoint summands $\{K^n\}_{n=1}^{\infty}$ and let K_n be a subgroup of K^n such that $K_n = \sum_{i=1}^{\infty} C_i(p^n)$. Let $K_0 = \sum_{n=1}^{\infty} K_n$. There exists a natural homomorphism from K onto K_0 and a natural homomorphism from K_0 onto $L \oplus D$. Thus we have a homomorphism f from K onto $L \oplus D$ and we may assume it so chosen that $o(x) > o(f(x))$ for every $0 \neq x \in K$. Next let $M = \{x + f(x) : x \in K\}$. Then $M = \sum_{i=1}^{\infty} \langle x_i + f(x_i)\rangle$ where the x_i are the generators of the summands of $K = \sum_{i=1}^{\infty} \langle x_i \rangle$ by Lemma 5. We also have that $h_G(x_i + f(x_i)) = h_G(x_i) = 0$ since $o(x) > o(f(x))$ for all $x \in K$. The purity of M follows from an argument similar to that in obtaining the purity of S in Case I. Next we can embed K and M in basic subgroups of G, say B_K and B_M, respectively. Now obviously $G = K + M$ and hence $G = B_K + B_M$.

To prove the converse suppose that G is not strongly starred. This means that there exists m such that G_m is not starred. Thus in particular $r(G_m) = |G_m|$ since $r(G_m)$ is infinite. It is well known that $r(G_m) = r(p^m G)$ (since $G_m[p] = (p^m G)[p]$; [1 p. 98]). Now suppose that $G = B + C$ where B and C are basic in G. Then

$$|G_m| = |p^m G| = |p^m(B + C)| \leq |p^m B| + |p^m C| < |G_m| + |G_m| = |G_m|$$

is a contradiction. Thus if G is a sum of two of its basic subgroups, G is strongly starred and the theorem is proved.

An argument similar to the last part of the foregoing proof shows easily the following interesting theorem.

THEOREM 15. *A p-group G is generated by n basic subgroups for some n if and only if G is generated by two of its basic subgroups.*

An example of a p-group without elements of finite height which is strongly starred and not $\sum^1$-cyclic is a direct sum $2^{\aleph_0}$ copies of a standard $\overline{B}$.

3. Some interesting questions

To conclude this paper we mention one final theorem and pose a few questions. A Serre class of Abelian groups is a class closed under subgroups, homomorphic images and extensions. In this connection we describe the most general class of $\sum^2$-cyclic groups closed under subgroups and pose the question, *What is the most general Serre class of $\sum^2$-cyclic groups?*

Let C be a class of torsion-free groups with the property that a group G belongs to C if and only if every subgroup H of finite rank is free. Then with C defined in this way we have

THEOREM 16. *The most general class $\mathfrak{G}$ of $\sum^2$-cyclic groups closed under subgroups is given by G belongs to $\mathfrak{G}$ if and only if*

(a) *G is reduced,*

(b) *G/T belongs to C.*

PROOF. This follows easily from Theorem 11 and the fact that a divisible group of rank 1 is not $\sum^2$-cyclic.

To specialize the above question concerning Serre classes of $\sum^2$-cyclic Abelian groups, we pose the question *Characterize the extensions of a $\sum^1$-cyclic by $\sum^1$-cyclic.* This question seems non-trivial even for primary groups.

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EINIGE BEMERKUNGEN ÜBER DIE AUTOMORPHISMEN ABELSCHER p-GRUPPEN

Von

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A. D. Wallace zum 60. Geburtstag

Von H. Freedman [1] stammt das folgende schöne Resultat: G sei eine reduzierte Abelsche p-Gruppe, S der Sockel von G, $S^{\mu} = S \cap (p^{\mu} G)$ die Ulm-Kaplanskysche Untergruppe aller Elemente aus S, deren Höhe in G mindestens gleich p^{μ} ist und $F^{\mu} = S^{\mu}/S^{\mu+1}$. Ein Endomorphismus η von G induziert dann in F^{μ} einen Endomorphismus η_{μ}.

SATZ. *Sei $p \geqq 5$ und G vom endlichen Ulmschen Typ, d. h. $p^{n\omega} G = 0$ für ein natürliches n. Ist dann a ein Automorphismus von G mit $a_{\mu} = 1$ für alle μ und $a^{p} = 1$, so ist a gleich 1 auf pG.*

Offenbar bilden die Automorphismen a von G mit $a_{\mu} = 1$ für alle μ einen Normalteiler K in der Automorphismengruppe $A = A(G)$. Aus dem obigen Ergebnis erhält man dann leicht den Satz, daß die Menge aller Elemente endlicher Ordnung aus K einen Normalteiler N in K bilden, der aus genau allen $a \in K$ besteht, die 1 auf einer Untergruppe $p^{i} G$ sind, $i < \omega$, vorausgesetzt, daß $p \geqq 5$ und der Ulmsche Typ von G endlich ist.

Wir wollen hier zeigen, daß auf die Forderung der Endlichkeit des Ulmschen Typs der Gruppe G verzichtet werden kann, d. h. wir werden zeigen:

Ist G eine reduzierte p-Gruppe, $p \geqq 5$, und a ein Automorphismus von G mit $a_{\mu} = 1$ für alle $\mu < \omega$, so ist $a = 1$ auf pG dann und nur dann, falls $a^{p} = 1$ ist.

Wir gehen aus von einer Folge direkter Zerlegungen von G:

(1) $$G = B_1 \oplus B_2 \oplus \ldots \oplus B_k \oplus H_k$$

mit

$$B_i \simeq \sum Z(p^{i}),$$

$$H_k = B_{k+1} \oplus H_{k+1},$$

$$H_k[p^{k}] \subset pH_k.$$

Dann ist also $B = \sum B_i$ eine Basisuntergruppe von G und $\bigcap\limits_{k} H_k = p^{\omega} G = G'$ die erste Ulmsche Untergruppe von G.

Es ist leicht zu sehen, daß für einen Endomorphismus η die Bedingungen $\eta_i = 0$, $i < \omega$ gleichbedeutend sind mit den Relationen

$$\eta\, B_{i+1} \subset B_1 \oplus \ldots \oplus B_i \oplus pB_{i+1} \oplus pH_{i+1}\,. \tag{2}$$

Ist überdies $p\,\eta = 0$, so folgt aus $p \cdot \eta G = 0$:

$$\eta\, G \subset S \subset B_1 \oplus pH_1$$

$$\eta^2 G \subset \eta B_1 + p\eta H_1 = \eta B_1 \subset pG$$

$$\eta^3 G \subset \eta\, pG = 0,$$

d. h. $\eta^3 = 0$. Ist nun $\alpha = 1 + \eta \in K$ und $\alpha = 1$ auf pG, d. h. $p\eta = 0$, so folgt:

$$\alpha^p = 1 + p\,\eta + \binom{p}{2}\eta^2 + \binom{p}{3}\eta^3 + \ldots = 1$$

falls $p \geqq 3$. Damit ist die Notwendigkeit von $\alpha^p = 1$ gezeigt, sogar für $p \geqq 3$.

Nun sei umgekehrt $\alpha = 1 + \vartheta \in K$, $\alpha^p = 1$. Für ϑ haben wir dann

$$p\,\vartheta + \binom{p}{2}\vartheta^2 + \ldots + p\,\vartheta^{p-1} + \vartheta^p = p\,\vartheta\beta + \vartheta^p = 0\,, \tag{3}$$

dabei ist

$$\beta = 1 + \frac{p-1}{2}\vartheta + \ldots + \vartheta^{p-2}$$

ein Automorphismus, denn es ist $1 - \beta = \vartheta\lambda$ mit einem mit ϑ vertauschbaren Endomorphismus λ. Aus (3) folgt $\vartheta^p = p\sigma$, also $(\vartheta\lambda)^p = p\tau$, τ mit $\vartheta\lambda$ vertauschbar. Folglich konvergiert $\sum\limits_{i=0}^{\infty} (\vartheta\lambda)^i = \beta^{-1}$ in der Topologie der punktweisen Konvergenz.

Das folgende Lemma enthält den wichtigsten Teil des Beweises unseres Satzes.

Lemma. *Sei $p \geqq 5$ und $1 + \vartheta \in K$. Ist dann $p^2\vartheta = 0$, so auch $p\vartheta = 0$.*

Beweis. Aus $p^2\vartheta = 0$ folgt zunächst

$$\vartheta\, pG \subset S \cap pG = pB_2 \oplus p^2H_2\,,$$

also $\vartheta^2 pG \subset p\vartheta\, B_2 \subset p^2H_2$ und somit $\vartheta^3 pG = 0$, d. h. $p\vartheta^3 = 0$.

Nun ist

$$\vartheta G \subset B_1 \oplus B_2 \oplus pB_3 \oplus p^2H_3 \tag{4}$$

Wir behaupten: $\vartheta^5 = 0$. Wegen (4) und $p\vartheta^3 = 0$ genügt es zu zeigen, daß $\vartheta^3\, B_1 = \vartheta^4\, B_2 = 0$ ist.

Aus $\vartheta B_1 \subset pB_2 \oplus p^2H_2$ folgt $\vartheta^2 B_1 \subset p\vartheta B_2 \subset p^2G$, also $\vartheta^3 B_1 = 0$. Aus $\vartheta B_2 \subset B_1 \oplus pH_1$ folgt schließlich $\vartheta^4 B_2 \subset \vartheta^3 B_1 + p\vartheta^3 H_1 = 0$. Damit ist

$\vartheta^5 = 0$ gezeigt. Insbesondere ist $\vartheta^p = 0$ falls $p \geqq 5$, aus (3) folgt deshalb $p\vartheta\beta = 0$ und $p\vartheta = 0$.

Wir können nun den oben formulierten Satz leicht beweisen. Sei λ die Länge von G, d. h. die kleinste Ordinalzahl mit $p^\lambda G = 0$. Für $\lambda \leqq \omega$ hat H. Freedman gezeigt, daß der Satz richtig ist, wir dürfen deshalb $\lambda > \omega$ annehmen und voraussetzen, daß die Behauptung für alle reduzierten Gruppen stimmt, falls deren Länge kleiner als λ ist.

Sei $\omega \leqq \mu < \lambda$ und $G_\mu = G/p^\mu G$. Dann hat G_μ die Länge μ und ϑ induziert in G_μ einen Endomorphismus ϑ', für den wieder die Voraussetzung $\vartheta'_i = 0$ für $i < \omega$ gilt. Das folgt daraus, daß wir wegen $p^\mu G \subset H_k$ für alle k die Zerlegung (1) für G_μ erhalten, indem wir dort H_k durch $H_k/p^\mu G = H'_k$ ersetzen und beachten, daß aus (2) dann für ϑ' die entsprechende Formel in G_μ folgt. Somit gilt nach Induktionsvoraussetzung $p\vartheta' = 0$, d. h. $p\vartheta G \subset p^\mu G$ für alle $\mu < \lambda$.

Ist λ Limeszahl, so folgt $p\vartheta G = 0$, also $p\vartheta = 0$. Ist $\lambda = \varkappa + 1$, so folgt insbesondere $p\vartheta G \subset p^\varkappa G$, also $p^2\vartheta G \subset p^{\varkappa+1} G = 0$, $p^2\vartheta = 0$ und deshalb wegen des Lemmas $p\vartheta = 0$. Damit ist der Satz bewiesen.

Es folgt nun leicht:

Die Menge T aller Automorphismen aus K von endlicher Ordnung, ist ein Normalteiler in K und in $A(G)$. Es ist $T = \bigcup_{i=1}^{\infty} T_i$, wobei T_i aus allen $\alpha \in K$ mit $\alpha^{p^i} = 1$ besteht. Es ist $\alpha \in T_i$ dann und nur dann, wenn $\alpha \in K$ und $\alpha = 1$ auf p^iG ist.

Zum Beweis ist im wesentlichen nur zu zeigen, daß für $\alpha \in K$ die Bedingungen $\alpha^{p^i} = 1$ und $\alpha = 1$ auf p^iG äquivalent sind. Für $i = 1$ hatten wir das bereits gezeigt. Nun wenden wir Induktion nach i an und setzen voraus, daß für $j < i$, $\alpha \in K$, die Bedingungen $\alpha^{p^j} = 1$ und $p^j\vartheta = 0$ äquivalent sind. Sei $\alpha^{p^i} = 1$. Es folgt, daß $\alpha^p = 1$ auf $p^{i-1}G = H$ und folglich $\alpha = 1$ auf $pH = p^iG$ ist, denn offensichtlich erfüllt die Einschränkung von α auf H wieder alle notwendigen Bedingungen bezüglich H.

Sei umgekehrt $p^i\vartheta = 0$. Es folgt $\alpha^{p^{i-1}} = 1$ auf pG, also $\alpha^{p^i} = (\alpha^{p^{i-1}})^p = 1$ auf G. Damit ist der Satz bewiesen.

Daß der Satz in seiner obigen scharfen Form für $p = 3$ falsch wird, zeigt das folgende Beispiel: Sei

$$G = (b_1) \oplus (b_2) \oplus (b_3)$$

mit $(b_i) \cong Z(3^i)$, d. h. b_i habe die Ordnung 3^i, $i = 1,2,3$. Jedem $\alpha \in A(G)$ ist die durch $\alpha b_i = \sum_{j=1}^{3} a_{ij}b_j$, $i = 1,2,3$ definierte Matrix (a_{ij}) zugeordnet. Dann definiert die Matrix $\begin{pmatrix} 1 & 3 & 0 \\ -1 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$

einen Automorphismus $\alpha = 1 + \vartheta$ aus K mit $\alpha^3 = 1$, jedoch

$$3\vartheta = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 9 \\ 0 & 0 & 0 \end{pmatrix} \neq 0 .$$

Es ist ein offenes Problem, ob für $p = 3$ aus $\alpha \in K$, $\alpha^3 = 1$ die Existenz eines natürlichen i mit $\alpha = 1$ auf $p^i G$ folgt.

Wir schließen mit einigen Bemerkungen zur Struktur der Automorphismengruppe $A = A(G)$. Man vergleiche hierzu auch die Arbeit von Fuchs [2].

Sei $\mathscr{E}$ der Endomorphismenring von G; $A(G)$ ist also die multiplikative Gruppe von $\mathscr{E}$. Der Ring $\mathscr{E}$ ist ein vollständiger topologischer Ring, sowohl in der p-adischen Topologie (definiert durch die Ideale $p^i\mathscr{E}$ als Fundamentalsystem von 0-Umgebungen), als auch in der Topologie der einfachen Konvergenz. Es folgt, daß für $i = 1, 2, \ldots$

$$N_i = \{1 + p^i\eta \mid \eta \in \mathscr{E}\}$$

ein Normalteiler in A ist.

Es ist $N_1 \supset N_2 \supset \ldots$ und $\bigcap N_i = \{1\}$. Ferner ist N_i/N_{i+1} Abelsch vom Exponenten p, nämlich isomorph zum Faktormodul $p^i\mathscr{E}/p^{i+1}\mathscr{E}$. Ein Automorphismus $\alpha = 1 + \vartheta$ liegt dann und nur dann in N_1, wenn $\vartheta S = 0$ und $\vartheta G \subset pG$ gilt, d.h. also wenn α in G/pG und in $S = G[p]$ den identischen Automorphismus induziert. Allgemeiner gilt:

Für $\eta \in \mathscr{E}$ sind die Bedingungen

1) $$\eta G \subset p^i G, \quad \eta G[p^i] = 0 ,$$

2) $$\eta \in p^i \mathscr{E} ,$$

äquivalent.

Was läßt sich über die Faktorgruppe A/N_1 sagen? Sei $F = G/pG$ und $S = G[p]$. Für $\alpha \in A$ sei α_F, bzw. α_S der von α in F, bzw. S induzierte Automorphismus. F und S sind A-Moduln und S besitzt die zu Beginn definierte Folge $\sum = \{S^\mu\}$ als Reihe A-invarianter Untermoduln. Ist $p^\omega G = 0$, d. h. enthält G keine Elemente unendlicher Höhe, so ist leicht zu sehen, daß $\sum$ eine Kompositionsreihe von S ist. Dasselbe gilt, wie H. Freedman gezeigt hat, falls G abzählbar ist. Der Normalteiler K aus A ist die zu $\sum$ gehörige Untergruppe aller $\alpha \in A$, die auf den Faktormoduln $S^\mu/S^{\mu+1} = F^\mu$, $S^\mu \in \sum$, gleich 1 sind. In F bilden die Moduln

$$F_i = (G[p^i] + pG)/pG$$

ebenfalls eine Reihe A-invarianter Untermoduln, diesmal mit $\bigcup F_i = F$. $\alpha \in K$ ist 1 auf allen $F_{i+1}/F_i = F_i^0$. Definieren wir nämlich

$$F_i^\mu = (p^\mu G \cap G[p^i])/(p^{\mu+1}G \cap G[p^i] + p^\mu G \cap G[p^{i-1}]),$$

so gilt folgender $\mathscr{E}$-Isomorphismus:

$$F_i^{\mu+1} \cong F_{i+1}^{\mu}.$$

Insbesondere sind also F_i^0 und $F_1^{i-1} = F^{i-1} = S^{i-1}/S^i$ A-isomorph.

Wir nennen eine Gruppe N p-nilpotent, wenn N eine wohlgeordnete Folge von Normalteilern N_μ enthält, so daß die folgenden Bedingungen erfüllt sind.

(a) $N = N_0$, $\bigcap N_\mu = \{1\}$,

(b) $N_\mu/N_{\mu+1}$ ist Abelsch vom Exponenten p.

Dann gilt:

Der Normalteiler K ist p-nilpotent.

Das ergibt sich aus der Tatsache, daß Untergruppen p-nilpotenter Gruppen wieder p-nilpotent sind, weiter aus den Eigenschaften von N_1 und dem folgenden leicht zu beweisenden Satz:

Sei $\mathfrak{M}$ eine Abelsche Gruppe vom Exponenten p, d. h. ein Modul über dem Galoisfeld $GF(p)$. Sei $\{\mathfrak{M}_\mu\}$ eine wohlgeordnete Folge von Untermoduln mit $\mathfrak{M}_0 = \mathfrak{M}$, $\bigcap\limits_\mu \mathfrak{M}_\mu = 0$, und sei N die Gruppe aller Automorphismen α von $\mathfrak{M}$ mit $\alpha\,\mathfrak{M}_\mu \subset \mathfrak{M}_\mu$, $\alpha = 1$ auf $\mathfrak{M}_\mu/\mathfrak{M}_{\mu+1}$. Dann ist N p-nilpotent.

Eine endliche p-nilpotente Gruppe ist eine p-Gruppe, insbesondere sind die Ordnungen der Elemente p-nilpotenter Gruppen Potenzen von p oder unendlich.

Ist G abzählbar oder ohne Elemente unendlicher Höhe, so ist A/K direktes oder subdirektes Produkt voller linearer Gruppen über $GF(p)$-Moduln, in diesen Fällen ist K also maximal p-nilpotent in A.

Im allgemeinen Fall ist nicht viel über die Struktur von A/K bekannt. Hier kann man einen kleinen Schritt weiter gehen und K durch eine u. U. größere Gruppe $\tilde{K}$ ersetzen, die wie folgt definiert ist:

Sei $\tilde{\sum}$ eine maximale Kette A-invarianter Untermoduln von S mit $\sum \subset \tilde{\sum}$. Im allgemeinen wird $\tilde{\sum}$ nicht mehr wohlgeordnet sein. Sei dann $\tilde{K}$ die Menge aller $\alpha \in A$, für welche gilt: Zu jedem $U \in \tilde{\sum}$ existiert ein $V \in \tilde{\sum}$ mit $V \subset U$, $U/V \neq 0$, $\alpha = 1$ auf U/V. Es läßt sich dann zeigen, daß $\tilde{K}$ ein fast-p-nilpotenter Normalteiler ist, im Sinne folgender Definition: Die Gruppe M heißt fast-p-nilpotent, wenn M ein System Ω von Normalteilern enthält mit folgenden Eigenschaften:

1) $M = \bigcup\limits_{N \in \Omega} N$.

2) Zu $N_1, N_2 \in \Omega$ existiert $N_3 \in \Omega$ mit $N_1 N_2 \subset N_3$.

3) Alle $N \in \Omega$ sind p-nilpotent.

Mir ist nicht bekannt, ob es Fälle gibt, in denen $\tilde{K}$ echt größer als K ist. Da außerdem $\tilde{K}$ keine Invariante von G ist, die Definition hängt von der Wahl einer maximalen Kette $\tilde{\sum}$ ab, scheint mir die Untersuchung von K zunächst die wichtigere Aufgabe zu sein.

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A-ORDERING OF THE GROUP EXT (B, A)

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Introduction

The Abelian extensions $G(A)$ of a (fixed) group A can be ordered in a natural way; the A-ordering of these extensions induces a so-called A-similarity and implies a partially ordered system $V(A)$ with a maximal and minimal element and with lattice properties. The analogous case is the B-ordering of the Abelian extensions $G(\alpha)$ through a (fixed) Abelian group B (denoting by α a homomorphism of the group G onto B); these extensions also give rise to a partially ordered system $P(B)$ with maximal and minimal elements. The general case—if A and B are not necessarily commutative—was considered in [3]. We now consider (for fixed Abelian groups A and B) the group $\mathrm{Ext}(B, A)$ of the equivalence classes of extensions $G(A, B)$ of A by B. As the groups $G(A, B)$ are extensions of A by B, the set of elements of Ext (B, A) must be a partially ordered subset of $V(A)$ and of $P(B)$. In this paper we investigate the A-ordering of Ext (B, A) in the case that A and B are finite groups. It will appear that this case is reduced to the case that A and B are cyclic groups of prime power order for the same prime number p.

§ 1

We consider the system of all the Abelian extensions of an (Abelian) group A. Denoting these extensions by $G(A)$, we define an A-ordering relation [1] as

$$G_1(A) \underset{A}{<} G_2(A) \, ,$$

(or $G_1(A) < G_2(A)$ if there is no possibility of confusion) if and only if there is an A-homomorphism η: $G_1 \rightarrow G_2$ (a homomorphism η of G_1 into G_2 leaving all the elements of A invariant). Defining $G_1(A) \underset{A}{\sim} G_2(A)$ if $G_1(A) < G_2(A)$ and $G_2(A) < G_1(A)$ (we say that: $G_1(A)$ and $G_2(A)$ are A-*similar*), any extension $G(A)$ defines a class $\{G(A)\}$, which consists of all the extensions of A similar to $G(A)$, and we obtain a partially ordered system $V(A)$ of classes $\{G(A)\}$; $\{G(A)\} < \{G'(A)\}$ only if this order relation holds for the two representatives $G(A)$ and $G'(A)$ of the corresponding classes. Any set of elements $\{G_i(A)\}$ has an A-meet and an A-join: the *A-meet* of the classes $\{G_i(A)\}$ is the class $\{G(A)\}$

$$\bigcap_i \{G_i(A)\} = \{G(A)\},\ G(A) = \sum_i{}^{*} G_i(A), i \in I.^{1}$$

Denoting the elements of $G(A)$ by $(g_i)_{i \in I}$ we see that G contains a subgroup $\cong A$, consisting of the elements $(a)_{i \in I}$. We have $G(A) < G_i(A)$, $i \in I$. Conversely, if $H(A) < G_i(A), i \in I$ and η_i: $H \to G_i$ are A-homomorphisms, then the mapping $h \to (h\, \eta_i)_{i \in I}$ is an A-homomorphism of H into $G(A)$. If $G_i(A)$, $i \in I$ is a set of extensions of A, the *A-union* of the corresponding classes $\{G_i(A)\}$ is constructed as follows: we consider the (restricted) direct sum $\sum_i G_i(A) = \sum$. This group $\sum$ contains a subgroup $\mathfrak{A}$ generated by the elements σ of the form

$$\sigma = (\ldots, 0, \ldots, 0, -a, 0 \ldots);$$

that are elements σ of $\sum$ containing a component a and a component $-a$ and the other components equal to 0. Therefore $\mathfrak{A}$ is the subgroup of $\sum$ of the elements $(\ldots, 0, a_1, \ldots, a_2, \ldots, a_n, 0, 0, \ldots)$ with $a_1 + a_2 + \ldots + a_n = 0$. We now consider the factor group $\mathfrak{G}(A) = \sum/\mathfrak{A}$; identifying the elements $a \in A$ with the classes $(0, \ldots, a, 0, \ldots)\, \mathfrak{A} = (a, 0, 0, 0, \ldots)\, \mathfrak{A}$, the mapping

$$a \to (0, \ldots, a, 0, \ldots)\, \mathfrak{A}$$

is an isomorphism of A with a subgroup of $\sum/\mathfrak{A}$. Hence $\mathfrak{G}(A)$ is an extension of A. $\mathfrak{G}(A)$ has the property of a union: if we map η_i: $g_i \to (0, 0, \ldots, g_i, \ldots)\mathfrak{A}$, $i \in I$, this mapping η_i is an A-homomorphism. If conversely $G_i(A) < H(A)$ with corresponding A-homomorphisms η_i, then $\mathfrak{G}(A) < H(A)$: we map

$$\eta\colon g = (\ldots, g_1, \ldots, g_j, \ldots, g_n, \ldots)\, \mathfrak{A} \to (g_1\eta_1) + \ldots + (g_n\eta_n).$$

Since H is Abelian and since this product is independent of the way we represent $g = (\ldots, g_1, \ldots, g_n, \ldots)\, \mathfrak{A}$, we see that η is a homomorphism, leaving the elements $a \in A$ invariant.

The minimal class $\{A\}$ of $V(A)$ consists of the extensions $G(A)$ having A as direct summand; for $G(A) < A$ implies $G = A + B$, if B is the kernel K_η of the A-homomorphism η: $G(A) \to A$. The maximal class $\{D(A)\}$ of $V(A)$ is defined by a divisible extension $D(A)$ of A. Using the fact that A can be embedded in a divisible group $D(A)$ and the fact that the identical mapping of $A \subseteq G(A)$ onto $A \subseteq D(A)$ can be extended to an A-homomorphism of $G(A)$ into $D(A)$, we see that $\{D(A)\}$ is the maximal class.

[1] $\sum$ denotes the restricted direct sum, $\sum^{*}$ the complete direct sum.

Now we consider the system of all the (Abelian) extensions by an (Abelian) group B, denoted by $G(\alpha)$; α is a homomorphism of G onto B. We define a B-order relation

$$G(\alpha) \underset{B}{<} G'(\alpha')$$

if and only if there is a homomorphism

$$\eta: G'(\alpha') \to G(\alpha)$$

such that $\eta\alpha = \alpha' \cdot G(\alpha)$ and $G'(\alpha')$ are called *B-similar* extensions (denoted by $G(\alpha) \underset{B}{\sim} G'(\alpha')$) only if

$$G(\alpha) < G'(\alpha') \text{ and } G'(\alpha') < G(\alpha) .$$

This B-similarity defines a partition of the system of the extensions by B into classes $\{G(\alpha)\}$ of B-similar extensions. The system $P(B)$ of these classes has the property that any set of classes has a B-join and a B-meet. The *B-join* $\bigcup_i \{G_i(\alpha_i)\}$ is the class $\{G(\alpha)\}$ of the subdirect sum $G(\alpha)$ with elements

$$g = (\ldots, g_i, \ldots, g_j, \ldots); \; g_i\alpha_i = \ldots = g_j\alpha_j = \ldots = g\alpha.$$

The *B-meet*

$$\bigcap_i \{G_i(\alpha_i)\} = \{H(\beta)\}$$

is the class of the direct sum $H(\beta) = \sum_i G_i(\alpha_i)(\beta)$ with

$$h\beta = (g_{i_1} + g_{i_2} + \ldots + g_{i_k})\beta = (g_{i_1}\alpha_{i_1}) + \ldots + (g_{i_k}\alpha_{i_k}) \in B.$$

The minimal class $\{B(\varepsilon)\}$ is the class of all groups having B as a direct summand. The maximal class is the class $\{F(\varrho)\}$ represented by a free Abelian group F having B as homomorphic image (see [3]).

§ 2

The set of the extensions $G(A, B)$ of A by B is a group $F(A, B)$ if we define $G_1(A, B) \dot{+} G_2(A, B) = G(A, B)$ as follows [2, p. 237]: if we have $G_i(A, B) = \dot{G}_i(\alpha_i)$, $i = 1, 2$, where α_i is the canonical homomorphism of G_i onto B,

$$G(A, B) = G_1 \overset{\times}{\underset{B}{\times}} G_2/\mathfrak{A} ,$$

where $G_1 \overset{\times}{\underset{B}{\times}} G_2$ is the subdirect sum of G_1 and G_2 consisting of all pairs (g_1, g_2) with $g_1\alpha_1 = g_2\alpha_2$, and $\mathfrak{A}$ is the subgroup of the pairs $(a, -a)$. The elements $a \in A$ correspond to the elements $(a, 0)\,\mathfrak{A}$ of G; writing $G(A, B) = G(\alpha)$ we have

$$((g_1, g_2)\,\mathfrak{A})\,\alpha = g_1\alpha_1 = g_2\alpha_2.$$

Defining two extensions $G(A, B)$ and $G'(A, B)$ equivalent, $G(A, B) \approx \approx G'(A, B)$ if there exists an isomorphism between them, leaving the elements of A and of B invariant, we know that the set of the extensions $G(A, B)$, which are equivalent to the direct sum $A + B$ of A and B, is a subgroup $T(A, B)$ of $F(A, B)$ and

$$\mathrm{Ext}\,(B, A) = F(A, B)/T(A, B)\,.$$

If $G_1(A, B)$ is defined by an extension factor system $\{f_1(b, b')\}$, and $G_2(A, B)$ by a factor system $\{f_2(b, b')\}$, the sum $G_1(A, B) \dot{+} G_2(A, B)$ can be defined by the factor system

$$\{f(b, b')\} = \{f_1(b, b') + f_2(b, b')\}\,.$$

$T(A, B)$ is the subgroup of the extensions that can be defined by the transformation systems $\{f(b, b')\}$ with

$$f(b, b') = c(b) + c(b') - c(b + b'),\; c(b) \in A.$$

From the definition of the equivalence of extensions it follows that equivalent extensions $G(A, B)$ and $G'(A, B)$ are A-similar and B-similar. This means that the extensions of one equivalence class of Ext (B, A) belong to one element of $V(A)$ and to one element of $P(B)$. It is therefore interesting to seek for the A-structure of Ext (B, A) (that means the embedding of Ext (B, A) in $V(A)$) and for the B-structure of Ext (B, A) (that is the embedding of Ext(B, A) in $P(B)$). When, in the following, we speak of the A-ordering (and of A-similarity) of two extensions $G_1(A, B)$ and $G_2(A, B)$, we mean the A-ordering etc. of G_1 and G_2 considered as extensions of A.

In this paper we shall speak only of the A-ordering of Ext (B, A) for finite (Abelian) groups A and B.

The A-join (resp. A-meet) of two extensions will be denoted by $G_1 \underset{A}{\cup} G_2$ (resp. $G_1 \underset{A}{\cap} G_2$) etc.

Theorem 1. *For any two extensions $G_1(A, B)$, $G_2(A, B)$ we have*

1) $G_1 \dot{+} G_2 \underset{A}{\leq} G_1 \underset{A}{\cup} G_2$,

2) $G_1 \underset{B}{\leq} G_2$ *implies* $G_1 \dot{+} G \underset{B}{\leq} G_2 \dot{+} G$ *for any extension* $G(A, B)$,

3) $G_1 \underset{B}{\sim} G_2$ *implies* $G_1 \dot{+} G \underset{B}{\sim} G_2 \dot{+} G$ *for any extension* $G(A, B)$,

4) *if* $G_1 \underset{A}{\leq} G_2$, $G_2 \underset{B}{\leq} G_1$ *holds for the same homomorphism* η: $G_1 \to G_2$ *we have* $G_1 \approx G_2$.

Proof. The mapping $(g_1, g_2)\,\mathfrak{A} \to (g_1, g_2)\,\mathfrak{A}$ proves 1); if η: $g_2 \to g_1$

satisfies $\eta\alpha_1 = \alpha_2$, then the mapping $(g_2, g)\,\mathfrak{A} \to (g_2\eta, g)\,\mathfrak{A}$ proves 2); 3) follows from 2); 4) follows from the commutativity of the diagram

$$\begin{tikzcd}[row sep=small] & G_1 \arrow[dd] \arrow[dr, "\alpha_1"] & & \\ 0 \arrow[r] & A \arrow[u, phantom] & & B \arrow[r] & 0 \\ & G_2 \arrow[ur, "\alpha_2"'] & & \end{tikzcd}$$

$$0 \to A \begin{array}{c} \overset{i_1}{\nearrow} G_1 \overset{\alpha_1}{\searrow} \\ \downarrow \\ \underset{i_2}{\searrow} G_2 \underset{\alpha_2}{\nearrow} \end{array} B \to 0.$$

§ 3

We want to consider the A-structure of Ext (B, A) in the case that A and B are finite groups. As B is a direct sum of cyclic groups B_i of prime power order, we have

$$\text{(1)} \qquad \operatorname{Ext}(B_1 + \ldots + B_k, A) \cong \operatorname{Ext}(B_1, A) + \ldots + \operatorname{Ext}(B_k, A).$$

Any system of extensions $G_i(A, B_i)$, $i = 1, \ldots, k$, of A defines in a unique way an extension $G(A, \sum_i B_i)$ of A by $\sum_i B_i$:

$$G(A, \sum_i B_i) = G_1 \underset{A}{\cup} G_2 \ldots \underset{A}{\cup} G_k = \sum_i G_i/\mathfrak{A}$$

with elements $(g_1, \ldots, g_k)\,\mathfrak{A}$, $g_i \in G_i$ (see § 1).

Choose in G_i the representatives $r^{(i)}(b_i)$ such that

$$r^{(i)}(b_i) + r^{(i)}(b'_i) = r^{(i)}(b_i + b'_i) + f^{(i)}(b_i, b'_i), \quad i = 1, \ldots, k,$$

and take for the respresentatives in G the elements $(r^{(1)}(b_1), \ldots, r^{(k)}(b_k))$; then the factor system in G, induced by the given choice of factor systems in G_i is

$$\{(f^{(1)}(b_1, b'_1), \ldots, f^{(k)}(b_k, b'_k))\,\mathfrak{A}\} = \{(\sum_i f^{(i)}(b_i, b'_i), 0, \ldots, 0)\,\mathfrak{A}\}.$$

$G(A)$ therefore can be constructed with the factor system $\{\sum_i f^{(i)}(b_i, b'_i)\}$ As G is an extension of A by $\sum_i B_i$ we denote this factor system by

$$\text{(2)} \qquad \{f((b_1, \ldots, b_k), (b'_1, \ldots, b'_k))\}$$

with

$$f((b_1, \ldots, b_k), (b'_1, \ldots, b'_k)) = f^{(1)}(b_1, b'_1) + \ldots + f^{(k)}(b_k, b'_k).$$

If all factor systems $\{f^{(i)}(b_i, b'_i)\}$, $i = 1, \ldots, k$ are transformation systems the induced factor system (2) is a transformation system. Conversely, any extension $G(A, \sum_i B_i)$, defined by a factor system (2) induces a set of factor systems $f^{(1)}(b_1, b'_1) = f((b_1, 0, 0, \ldots, 0), (b'_1, 0, 0, \ldots, 0)), \ldots$ etc. and they define extensions $G_i(A, B_i)$, $i = 1, 2, \ldots, k$. These G_i are independent of the chosen factor system in $G(A, \sum_i B_i)$ and they induce (taking their A-union) the original extension $G(A, \sum B_i)$. That means: any extension $G(A, \sum_i B_i)$ is the A-union of a finite number of extensions $G_i(A, B_i)$, $i = 1, \ldots, k$. The used mapping gave us a way of proving the isomorphism (1). Therefore we have

THEOREM 2. *The A-structure of* $\mathrm{Ext}(\sum_i B_i, A)$ *follows from the A-structures of* $\mathrm{Ext}(B_i, A)$ *taking the A-unions of the elements of* $\mathrm{Ext}(B_i, A)$, $i = 1, \ldots, k$.

Now we have to consider the isomorphism

$$\mathrm{Ext}\,(B, A_1) + \ldots + \mathrm{Ext}\,(B, A_k) \cong \mathrm{Ext}\,(B, A_1 + \ldots + A_k).$$

By taking $k = 2$ we reduce the calculations. Suppose $G_1(A_1, B)$ is an extension defined by the factor system $\{f_1(b, b')\}$; $G_1(A_1, B)$ induces an extension $G_1(A_1 + A_2, B)$ of $A_1 + A_2$ by B, defined by the factor system

$$\{f(b, b')\} = \{(f_1(b, b'), 0)\};$$

$\{f(b, b')\}$ is a transformation system if and only if $\{f_1(b, b')\}$ is a transformation system.

If $G_1(A_1, B) \underset{A_1}{<} \bar{G}_1(A_1, B)$, then $G_1(A_1 + A_2, B) \underset{A_1+A_2}{<} \bar{G}_1(A_1 + A_2, B)$, for if

$$\eta\colon (b; a_1) \to (\bar{b}, \bar{a}_1)$$

is an A_1-homomorphism of G_1 into $\bar{G}_1$, we define an $(A_1 + A_2)$-homomorphism η' of $G_1(A_1 + A_2, B) \to \bar{G}_1(A_1 + A_2, B)$ as

$$\eta'\colon (b, a_1, a_2) \to (\bar{b}, \bar{a}_1, a_2).$$

Conversely, suppose that $G_1(A_1 + A_2, B)$ and $\bar{G}_1(A_1 + A_2, B)$ are images of $G_1(A_1, B)$, resp. $G_2(A_2, B)$, then $G_1(A_1 + A_2, B) \underset{A_1+A_2}{<} \bar{G}_1(A_1 + A_2, B)$ implies $G_1(A_1, B) \underset{A_1}{<} \bar{G}_1(A_1, B)$. This proves

THEOREM 3. *In $V(A_1)$ the A_1-structure of the extensions $G_1(A_1, B)$ induces in $V(A_1 + A_2)$ exactly the same $(A_1 + A_2)$-structure for the induced extensions $G_1(A_1 + A_2, B)$.*

§ 4

If A and B are finite groups, the A-structure of Ext (B, A) can be found by means of the results of § 3. If

$$A = \sum_{i=1}^{r} C^{(i)}(p_i^{n_i}), \quad B = \sum_{j=1}^{s} C^{(j)}(p_j^{n_j}),$$

we have

$$\mathrm{Ext}\,(B, A) \cong \sum_{i,j} \mathrm{Ext}\left(C^{(j)}(p_j^{n_j}),\ C^{(i)}(p_i^{ni})\right).$$

For a cyclic group $B = C(m)$ of order m we have

$$\mathrm{Ext}\,(B, A) \simeq A/mA.$$

If $p_i \neq p_j$ then

$$p_j^{n_j} \cdot C^{(i)}(p_i^{n_i}) = C^{(i)}(p_i^{n_i}).$$

Therefore

$$\mathrm{Ext}\,(B, A) \simeq \mathrm{Ext}\,(C(p_i^{m_i}), C(p_i^{n_i}))$$

and we have to investigate the A-ordering of the groups $\mathrm{Ext}\,(C(p^m), C(p^n))$. The extensions $G(A, B)$, where $A = C(p^n)$, $B = C(p^m)$, have the order p^{m+n} and the number of the inequivalent extensions of A by B is given by the order of

$$\mathrm{Ext}\,(C(p^m), C(p^n)) \simeq C(p^n)/p^m C(p^n) \simeq C(p^{\min(m,n)}).$$

The number of non-isomorphic groups G of order p^{m+n} is equal to the number of partitions of $m+n$ into positive integers. We have, however, the condition that G must contain a subgroup $C(p^n)$, and that the factor group $G/C(p^n) \simeq C(p^m)$. We shall denote the extensions $G(C(p^n), C(p^m))$ by the *type*

$$\begin{pmatrix} p^{m_1}, & p^{m_2}, & \ldots, & p^{m_k} \\ p^n, & 1 & , \ldots, & 1 \end{pmatrix}, \quad m_1 \geqq m_2 \geqq \ldots \geqq m_k\,,$$

where the first row $(p^{m_1}, \ldots, p^{m_k})$ denotes the type of G, the second row $(p^n, 1, \ldots)$ the type of A; we have $m_1 + m_2 + \ldots + m_k = m + n$.

THEOREM 4. *The number of inequivalent extensions of type* $\begin{pmatrix} p^{m+n} \\ p^n \end{pmatrix}$ *is* $(p-1).\, p^{\min(m,n)-1}$; *these extensions are all A-similar (and B-similar).*

PROOF. We can embed $C(p^n)$ in $C(p^{m+n})$ in $\varphi(p^n) = p^n - p^{n-1}$ different ways and that gives rise to $p^{n-1}(p-1)$ extensions of type $\begin{pmatrix} p^{m+n} \\ p^n \end{pmatrix}$. To find the number of equivalent extensions among them we have to find the automorphisms of $C(p^n)$, inducing an endomorphism of $C(p^{m+n})$, that leaves all the cosets invariant. Therefore we map $p^m \to (qp^m + 1)\, p^m < p^{m+n}$ or $qp^m + 1$. If $m \geqq n$ then for q there is no solution (only the identity); in this case we have already $p^{n-1}(p-1)$ inequivalent extensions G of type $\begin{pmatrix} p^{m+n} \\ p^n \end{pmatrix}$. If $m < n$, then we have p^{n-m} possibilities for q; that means that we have

$$\frac{p^{n-1}(p-1)}{p^{n-m}} = p^{m-1}(p-1)$$

inequivalent extensions of type $\begin{pmatrix} p^{m+n} \\ p^n \end{pmatrix}$.

THEOREM 5. *For an extension G of $A = C(p^n)$ by $B = C(p^m)$ of type* $\begin{pmatrix} p^{m_1}, p^{m_2}, \ldots, p^{m_k} \\ p^n, 1, \ldots, 1 \end{pmatrix}$, $m_1 = \geqq \ldots \geqq m_k$, $\sum m_i = m + n$, *we have:*

1) $k \leqq 2$,

2) *if the type of G is* $\begin{pmatrix} p^{m_1}, p^{m_2} \\ p^n, 1 \end{pmatrix}$, *then* $m_1 \geqq m$, $m_1 \geqq n \geqq m_2$,

3) *all these types represent extensions of A by B.*

PROOF. 1) Suppose $G(\alpha)$ is an extension of $A = C(p^n)$ by $B = C(p^m)$ with the canonical homomorphism α: $G \to B$. If $A = \{a\}$ and $g \in G$ such that $B = \{g\,\alpha\}$, then $G = \{a, g\,\alpha\}$; therefore G can never have the mentioned type with $k > 2$.[2] 2) Suppose $m_1 \geqq m_2 > 0$ then we must have $m_1 \geqq n$. Suppose $C(p^{m_1}) = \{a_1\}$, $C(p^{m_2}) = \{a_2\}$, then $p^{m_1}(a_1, a_2) = (0, 0)$. As the factor group B must have the order p^m we must have $m_1 \geqq n$. Therefore $m_1 + m_2 \geqq m + m_2$ or $m + n < m + m_2$, hence $n \geqq m_2$. 3) If $m_2 = 0$, then $G = C(p^{m+n})$ is an extension of $C(p^n)$ by $C(p^m)$. Suppose $m_2 > 0$; if $m_1 = n$ (or $= m$) then $G = C(p^n) \times C(p^m)$ satisfies all conditions. Therefore we consider only $m_1 > n$, $m_1 > m$. This implies $m_1 > n > m_2$ and $m_1 > m > m_2$. We take for A the subgroup generated by $(p^{m_1-n}, 1)$. If we take the coset of A in G containing the element $(1, 1)$, we have $p^{m_2}(1, 1) = (p^{m_2}, 0)$, but this is no element of A because $(p^{m_1+m_2-n}, 0) = (p^m, 0)$ is the smallest multiple of $(p^{m_1-n}, 1)$ with second component equal to zero. Therefore the group G of the type (p^{m_1}, p^{m_2}), $m_1 > n > m_2$, $m_1 > m > m_2$ contains a subgroup $A = C(p^n)$ with $G/A \cong C(p^m)$.

THEOREM 6. *Two extensions G and G' of $A = C(p^n)$ by $B = C(p^m)$ of the same type* $\begin{pmatrix} p^{m_1}, p^{m_2} \\ p^n, 1 \end{pmatrix}$ *are A-similar.*

PROOF. The extensions G and G' may differ in the way A is embedded. If A is generated by $(p^{m_1-n}, 0)$, we have for the type of G/A: (p^{m_1-n}, p^{m_2}). Therefore we generate A by $(p^{m_1-n}, 1)$; in this case the coset represented by $(0, 1)$ has the order p^m, that means $G/A \cong C(p^m)$. Suppose that A (in G') is generated by (ap^{m_1-n}, b), $(a, p) = 1$, $b \in C(p^{m_2})$, $b \neq 0$. The homomorphism of G into G' induced by the mapping

$$(p^{m_1-n}, 1) \to (ap^{m_1-n}, b), \quad (0, 1) \to (0, 1)$$

is an A-homomorphism of G into G'. In the same way one proves $G' \lesssim_A G$.

THEOREM 7. *If the extensions G resp. G' of $A = C(p^n)$ by $B = C(p^m)$ have the type* $\begin{pmatrix} p^{m_1}, p^{m_2} \\ p^n, 1 \end{pmatrix}$ *resp.* $\begin{pmatrix} p^{m_1'}, p^{m_2'} \\ p^n, 1 \end{pmatrix}$, *then these extensions satisfy the condition $G \lesssim_A G'$, if and only if $(m_1, m_2) < (m_1', m_2')$ in the lexicographical*

[2] The proof of the first statement of this theorem, given by G. Pollák (Szeged), is a simplification of my proof.

sense (that means: $(m_1, m_2) < (m_1', m_2')$ *if and only if* $m_1 < m_1'$ *or* $m_1 = m_1'$, $m_2 \leq m_2'$*).*

PROOF. Suppose $(m_1, m_2) < (m_1', m_2')$ and moreover that A is generated by $(p^{m_1-n}, 1)$ in G and by $(p^{m_1'-n}, 1)$ in G', then there is an A-homomorphism η: $G \to G'$ induced by

$$\eta\colon (1, 0) \to (p^{m_1'-m_1}, 0), \quad \eta\colon (0, 1) \to (0, 1)\,.$$

Conversely: suppose η: $G \to G'$ is an A-homomorphism of G into G'. We suppose that A in G is generated by $(p^{m_1-n}, 1)$, A in G' by $(p^{m_1'-n}, 1)$; then there must be an A-homomorphism mapping

$$(1, 0) \to (a, b), \ (0, 1) \to (a', b')$$

and inducing

$$(p^{m_1-n}, 1) \to (p^{m_1'-n}, 1)\,.$$

We can suppose that $m_1 > m_2$, for if $m_1 = m_2$ then we have $m_1 = m_2 = n$ and then we should have $m_1' = n = m_2'$ and the theorem is trivial. Now

$$(p^{m_1-n}, 1) \to (p^{m_1-n}\, a + a', p^{m_1-n}\, b + b')\,.$$

The element

$$p^{m_2}(p^{m_1-n}, 1) = (p^{m_1-n+m_2}, 0)$$

is the smallest multiple of the generating element of A with second component 0. Therefore $p^{m_2} \cdot p^{m_1-n}\, a = p^{m_1+m_2-n} \cdot a$ must be mapped onto $p^{m_1'-n+m_2}$, therefore

$$p^{m_1+m_2-n} \cdot a \equiv p^{m_1'-n+m_2} \pmod{p^{m_1'}}$$

or

$$p^{m_1'} \mid p^{m_1+m_2-n} \cdot a - p^{m_1'-n+m_2}\,;$$

if $m_1' = m + k$, $k \geqq 0$, we have

$$p^{m+k} \mid p^m \cdot a - p^{m+k-n+m_2}$$

and we must have (if $a \neq 0$) $k - n + m_2 \geqq 0$ or $m_1' \geqq m_1$ and therefore $m_2' \leqq m_2$ proving the theorem. If a $a = 0$ we should have $m_2 = n$, that means $m_2 = n$ and from $m_1 + m_2 = m + n$ it followed $m_1 = m$ contradicting our supposition $m_1 > m$. The last result shows that the A-ordering of the equivalence classes of Ext $(C(p^m), C(p^n))$ is a linear ordering.

EXAMPLE I. If $A = B = C(p)$ there are p inequivalent extension, $G(C(p), C(p))$; there is an A-minimal extension $G_1\,(A, B) = C(p) \times C(p)$s while the other $p - 1$ extensions $G_2\,(A, B), \ldots, G_p\,(A, B)$ are cyclic groups of order p^2. Mapping the element 1 on $2, 3, \ldots, p - 1$, resp. we can embed $C(p)$ in $p - 1$ different ways in $C(p^2)$; these $p - 1$ extensions are A-similar but not equivalent. The A-diagram is:

$$\begin{array}{l} G_2 \underset{A}{\sim} \cdots \underset{A}{\sim} G_p \ (\text{type } (p^2, 1)) \\ \big\uparrow \\ G_1 \ (\text{type } (p, p)) \end{array}$$

Example II. If $A = C(p^3)$, $B = C(p^2)$ the diagram of $\mathrm{Ext}\,(C(p^2), C(p^3))$ is:

$$\begin{array}{l} G_3 \text{ (type } (p^5, 1)) \\ \uparrow \\ G_2 \text{ (type } (p^4, p)) \\ \uparrow \\ G_1 \text{ (type } (p^3, p^2)) \end{array}$$

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ON THE STRUCTURE OF TOR[1]

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This paper is concerned with the structure of the torsion product Tor (A, B) of Abelian groups A and B. Since Tor(A, B) is the direct sum of the groups Tor(t_pA, t_pB) where t_pA and t_pB are the p-primary components of A and B, respectively, I shall confine my attention almost exclusively to p-primary groups.

The paper is divided into three sections. The first section is devoted to computing the Ulm invariants of Tor (A, B) in terms of the Ulm invariants of A and of B and other known invariants. In view of the results of Kolettis [3], Tor (A, B) in thus known whenever A and B are direct sums of countable groups.

In the second section I study the effect of certain restrictions on A and B on the structure of Tor (A, B). This investigation shows that there are p-primary groups without elements of infinite height which are not contained in any group of the form Tor (A, B) with A and B reduced.

Finally in the last section the results of the first two sections are applied to the study of generalized purity begun in [5]. Some questions left open in that paper are answered. For example it is shown that p^{α} Ext is a hereditary functor if α is a countable ordinal.

1. In order to compute the Ulm invariants of Tor (A, B) a number of lemmas will be needed.

LEMMA 1.1. *If n is an integer, then*

$$\text{Tor}\,(A, B)[n] = \text{Tor}\,(A[n], B) = \text{Tor}\,(A[n], B[n]).$$

PROOF. The second equality follows from the first and the symmetry of Tor. If Tor is applied to the exact sequence $0 \to A[n] \to A \xrightarrow{n} A$ where the map $A \xrightarrow{n} A$ is multiplication by n, the exact sequence

$$0 \to \text{Tor}\,(A[n], B) \to \text{Tor}\,(A, B) \xrightarrow{n} \text{Tor}\,(A, B)$$

results giving the first equality.

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A *subfunctor of the identity* is a function S which assigns to each group A a subgroup SA in such a way that if $f\colon A \to B$ is a homomorphism, then $fSA \subseteq SB$.

LEMMA 1.2. *If S is a subfunctor of the identity and $SA = 0$, then*

$$S \operatorname{Tor}(A, B) = 0.$$

PROOF. We can assume that B is a torsion group. Let K be the group of rational numbers modulo the integers. Then $B \subseteq \sum K$ where $\sum K$ is a direct sum of copies of K. Tor is left exact, and commutes with direct sums, and $\operatorname{Tor}(A, K) \cong A$. Hence $\operatorname{Tor}(A, B) \subseteq \sum A$. Each subfunctor of the identity also commutes with direct sums and $C \subseteq D$ implies $SC \subseteq SD$. Hence $S \operatorname{Tor}(A, B) \subseteq \sum SA = 0$.

LEMMA 1.3. *If S is a subfunctor of the identity such that $S(A/SA) = 0$ for all A, then*

$$S \operatorname{Tor}(A, B) \subseteq \operatorname{Tor}(SA, SB).$$

PROOF. Since Tor is left exact, the sequence

$$0 \to \operatorname{Tor}(SA, SB) \to \operatorname{Tor}(A, B) \to \operatorname{Tor}(A/SA, B) \oplus \operatorname{Tor}(A, B/SB)$$

is exact where the two components of the right hand map are induced by the maps $A \to A/SA$ and $B \to B/SB$ respectively. The lemma now follows from the hypothesis and lemma 1.2.

LEMMA 1.4. *If n is an integer, then*

$$n \operatorname{Tor}(A, B) = \operatorname{Tor}(nA, nB).$$

PROOF. The function sending A into nA is a subfunctor of the identity. Hence $n \operatorname{Tor}(A, B) \subseteq \operatorname{Tor}(nA, nB)$ follows from Lemma 1.3.

In proving the opposite inclusion it is convenient to use the definition of Tor given by MacLane [4]. $\operatorname{Tor}(A, B)$ is the Abelian group having as generators all symbols $\langle a, n, b\rangle$ with n an integer, $na = 0$ in A, and $nb = 0$ in B, subject to the relations

$$\begin{aligned}
&\langle a_1 + a_2, n, b\rangle = \langle a_1, n, b\rangle + \langle a_2, n, b\rangle && na_i = 0 = nb,\\
&\langle a, n, b_1 + b_2\rangle = \langle a, n, b_1\rangle + \langle a, n, b_2\rangle && na = 0 = nb_i,\\
&\langle a, nm, b\rangle = \langle na, m, b\rangle && nma = 0 = mb,\\
&\langle a, nm, b\rangle = \langle a, n, mb\rangle && na = 0 = nmb.
\end{aligned}$$

The first two relations imply that

$$n\langle a, m, b\rangle = \langle na, m, b\rangle = \langle a, m, nb\rangle.$$

Now suppose that $x = \langle na, m, nb \rangle$ is a generator of Tor (nA, nB). Then $nma = 0 = nmb$ so that

$$x = \langle a, nm, nb \rangle = \langle a, nm, b \rangle .$$

Hence x is in n Tor (A, B) as required.

If p is a prime and α is an ordinal number, then $p^{\alpha} A = p(p^{\beta+1} A)$ if $\alpha = \beta + 1$ and $p^{\alpha} A = \bigcap_{\beta > \alpha} p^{\beta} A$ if α is a limit ordinal. Then p^{α} is a subfunctor of the identity and $p^{\alpha} (A/p^{\alpha}A) = 0$.

THEOREM 1.5. *For p a prime and α an ordinal,*

$$p^{\alpha} \operatorname{Tor} (A, B) = \operatorname{Tor} (p^{\alpha}A, p^{\alpha}B).$$

PROOF. We proceed by induction on α. The case $\alpha = 0$ is trivial and the step from α to $\alpha + 1$ is made with the help of Lemma 1.4. Suppose α is a limit ordinal and the theorem is true for $\beta < \alpha$. The inclusion $p^{\alpha} \operatorname{Tor} (A, B) \subseteq \operatorname{Tor} (p^{\alpha} A, p^{\alpha} B)$ comes from Lemma 1.3. If $\beta < \alpha$, then $p^{\alpha} A \subseteq p^{\beta} A$ and $p^{\alpha} B \subseteq p^{\beta} B$ so that

$$\operatorname{Tor} (p^{\alpha} A, p^{\alpha} B) \subseteq \operatorname{Tor} (p^{\beta} A, p^{\beta} B) = p^{\beta} \operatorname{Tor} (A, B) .$$

It follows that

$$\operatorname{Tor} (p^{\alpha} A, p^{\alpha} B) \subseteq \bigcap_{\beta < \alpha} p^{\beta} \operatorname{Tor} (A, B) = p^{\alpha} \operatorname{Tor} (A, B).$$

It is worth noting that if A^{α} is the α-th Ulm subgroup of A defined by $A^{\alpha+1} = \bigcap_{n \neq 0} nA^{\alpha}$ and $A^{\alpha} = \bigcap_{\beta < \alpha} A^{\beta}$ for α a limit ordinal, then the method of proof just used gives

$$\operatorname{Tor} (A, B)^{\alpha} = \operatorname{Tor} (A^{\alpha}, B^{\alpha}).$$

For the rest of the paper p will be a fixed prime and all groups will be p-primary. If A is reduced, then the *length* $\lambda(A)$ of A is the least ordinal α such that $p^{\alpha} A = 0$.

COROLLARY 1.6. *If A and B are reduced p-groups, then* Tor (A, B) *is reduced and its length is the minimum of the lengths of A and of B.*

The *p-rank* of A, denoted by $r(A)$, is the dimension of $A[p]$ as a vector-space over the field of integers modulo p. The α-th *Ulm invariant* of A, denoted by $f_{\alpha} (A)$, is the dimension of $(p^{\alpha} A)[p]/(p^{\alpha+1}A)[p]$ over this same field. After writing $r_{\alpha} (A)$ for $r(p^{\alpha} A)$ we have the equality

$$r_{\alpha} (A) = r_{\alpha+1} (A) + f_{\alpha} (A).$$

THEOREM 1.7. *For any ordinal α,*

$$r_{\alpha} (\operatorname{Tor}(A, B)) = r_{\alpha}(A)\, r_{\alpha}(B)$$

and

$$f_{\alpha} (\operatorname{Tor}(A, B)) = f_{\alpha} (A)f_{\alpha} (B) + f_{\alpha} (A)r_{\alpha+1} (B) + r_{\alpha+1} (A)f_{\alpha} (B).$$

PROOF. If A is a cyclic group of order p, then Tor $(A, A) = A$. Tor also commutes with direct sums. It follows that if $pA = 0 = pB$, then $r(\mathrm{Tor}(A, B)) = r(A)r(B)$.

Theorem 1.5 and Lemma 1.1 give the equality

$$(p^a \,\mathrm{Tor}\,(A, B))[p] = \mathrm{Tor}\,((p^a A)[p], (p^a B)[p])$$

which together with the definition of r_a and the remarks just made prove the first equality of the theorem.

Since, for each group C, $C[p]$ is a vector space over the integers modulo p, every subgroup of $C[p]$ is a direct summand. Hence

$$(p^a A)[p] = (p^{a+1} A)[p] \oplus U$$

and

$$(p^a B)[p] = (p^{a+1} B)[p] \oplus V$$

where $U \cong (p^a A)[p]/(p^{a+1} A)[p]$ and $V \cong (p^a B)[p]/(p^{a+1} B)[p]$. Let us set $W = \mathrm{Tor}\,(A, B)$. Then $(p^a W)[p] = \mathrm{Tor}\,\big((p^a A)[p], (p^a B)[p]\big)$ as we saw in the last paragraph. Therefore

$$(p^a W)[p] = \mathrm{Tor}\,\big((p^{a+1} A)[p], (p^{a+1} B)[p]\big) \oplus \mathrm{Tor}\,(U, (p^{a+1} B)[p]) \oplus$$
$$\oplus\, \mathrm{Tor}\,\big((p^{a+1} A)[p], V\big) \oplus \mathrm{Tor}\,(U, V).$$

The first term on the right side of this equation is $(p^{a+1}W)[p]$ so that $(p^a W)[p]/(p^{a+1}W)[p]$ is isomorphic to the direct sum of the remaining three terms. Now the definitions of r_a, f_a, and the first paragraph of the proof give the theorem.

Using the fact that $r_a = r_{a+1} + f_a$ we can restate the second equation of the theorem as

$$f_a\,(W) = f_a\,(A) r_a\,(B) + r_{a+1}\,(A) f_a\,(B) = f_a\,(A) r_{a+1}\,(B) + r_a\,(A) f_a\,(B).$$

Note that $f_a\,(A) \leq f_a\,(W)$ and $f_a\,(W) = 0$ if and only if $f_a(A) = 0 = f_a\,(B)$.

COROLLARY 1.8. *If A and B are reduced countable p-groups, then A is isomorphic to a direct summand of* Tor (A, B) *if and only if $\lambda(A) \leqq \lambda(B)$.*

PROOF. Set $W = \mathrm{Tor}\,(A, B)$. Since $\lambda(W) = \min\,(\lambda(A), \lambda(B))$, the proof in the forward direction is clear.

Suppose $\lambda(A) \leq \lambda(B)$. Then $\lambda(W) = \lambda(A) = \lambda + n$ where λ is a limit ordinal and $n < \omega$. Since A and B are countable, we have $r_a\,(A) = \aleph_0 = r_a\,(B)$ whenever $a < \lambda$. It follows from the formula for $f_a\,(W)$ that

$$f_a\,(W) = \begin{cases} 0 & \text{if } f_a(A) = 0 = f_a(B), \\ \aleph_0 & \text{otherwise} \end{cases}$$

when $a < \lambda$. Define a sequence of cardinals h_a as follows: for $a < \lambda$ put $h_a = 0$ or $\aleph_0$ according as $f_a\,(W) = 0$ or $\aleph_0$; for $\lambda \leq a < \lambda(W)$ choose h_a

so that $f_\alpha(A) + h_\alpha = f_\alpha(W)$. Then $f_\alpha(A) + h_\alpha = f_\alpha(W)$ for all $\alpha < \lambda(W)$. Moreover $h_\alpha \neq 0$ for infinitely many values of α between any two limit ordinals less than $\lambda(W)$. Hence there is a countable reduced group C with $f_\alpha(C) = h_\alpha$ for all α [3]. Then $f_\alpha(A \oplus C) = f_\alpha(A) + f_\alpha(C) = f_\alpha(W)$ for all α. Hence Ulm's Theorem gives $A \oplus C \cong W$.

2. Since $A = \mathrm{Tor}\,(Z(p^\infty), A)$, any p-group is the torsion product of two p-groups. However not every group is isomorphic to a group of the form $\mathrm{Tor}\,(A, B)$ with A and B reduced.

For S a subfunctor of the identity, let $\mathscr{H}(S)$ be the class of all p-groups A such that

(i) $S(A) = 0$ and

(ii) each infinite subgroup B of A is contained in a subgroup C of the same power such that $S(A/C) = 0$.

THEOREM 2.1. *If S is a subfunctor of the identity and $S(A) = 0 = S(B)$, then* $\mathrm{Tor}\,(A, B) \in \mathscr{H}(S)$.

PROOF. Let G be an infinite subgroup of $\mathrm{Tor}\,(A, B)$. Since each element of G is a finite sum of elements $\langle a, n, b\rangle$ of $\mathrm{Tor}\,(A, B)$, there are subgroups A_0, V_0 of A and B respectively such that $G \subseteq \mathrm{Tor}\,(A_0, B_0)$ and $|\mathrm{Tor}\,(A_0, B_0)| = |G|$. By the left exactness of Tor there is an exact sequence $0 \to \mathrm{Tor}\,(A_0, B_0) \to \mathrm{Tor}\,(A, B) \to \mathrm{Tor}\,(A/A_0, B) \oplus \mathrm{Tor}\,(A, B/B_0)$. In view of Lemma 1.2 the condition $S(A) = 0 = S(B)$ implies that the right hand group in this sequence is annihilated by S, hence $\mathrm{Tor}\,(A, B)/\mathrm{Tor}\,(A_0, B_0)$ is also annihilated by S.

THEOREM 2.2. *Let S be a subfunctor of the identity.*

(i) *If $A \in \mathscr{H}(S)$ and $B \subseteq A$, then $B \in \mathscr{H}(S)$.*

(ii) *If $\{A_i\}$ is a family of groups in $\mathscr{H}(S)$, then $\sum A_i \in \mathscr{H}(S)$.*

(iii) *If $A \in \mathscr{H}(S)$, then* $\mathrm{Tor}\,(A, B) \in \mathscr{H}(S)$.

(iv) *If $A \in \mathscr{H}(S)$, then $S(A) = 0$ and, for every infinite subgroup B of A, $|S(A/B)| \leq |B|$.*

(v) *If $S(G/SG) = 0$ for all p-groups G, the converse of* (iv) *holds.*

PROOF. Part (i) is trivial. Suppose $A_i \in \mathscr{H}(S)$ and $B \in \sum A_i$ is infinite. Let π_i be the i-th coordinate projection. Since, for each b in B, $\pi_i b = 0$ for almost all indices, there is a set J of indices such that $|J| \leq |B|$ and $\pi_i B = 0$ for i not in J. We lose nothing in assuming that J contains all the indices. Since $A_i \in \mathscr{H}(S)$ there is a C_i such that $\pi_i B \subseteq C_i \subseteq A_i$, $|C_i| = |\pi_i B|\,\aleph_0 \leq |B|$ and $S(A_i/C_i) = 0$. Then $B \subseteq \mathscr{H}\,C_i$, $|B| = |\sum C_i|$ and $S(\sum A_i/\sum C_i) \cong \sum S(A_i/C_i) = 0$.

It was shown in the proof of Lemma 1.2 that $\mathrm{Tor}\,(A, B)$ is contained in a direct sum of copies of A. Hence (iii) follows from (i) and (ii).

Suppose $A \in \mathscr{H}(S)$ and B is an infinite subgroup of A. Then $B \subseteq C \subseteq$ $\subseteq A$ with $|C| = |B|$ and $S(A/C) = 0$. Since $0 \to C/B \to A/B \to A/C$ is exact, $S(A/B) \subseteq C/B$. Hence $|S(A/B)| \leq |C/B| \leq |C|$ proving (iv).

Now suppose that $S(G/SG) = 0$ for all p-groups G and that $|S(A/B)| \leq \leq |B|$ whenever B is an infinite subgroup of A. Let B be an infinite subgroup of A and let $C \subseteq A$ be such that $C/B = S(A/B)$. Since $A/C = = (A/B)/(C/B) = (A/B)/S(A/B)$, $S(A/C) = 0$. Moreover $|C| = |C/B| + |B| = = |S(A/B)| + |B| \leq |B| + |B| = |B|$.

The class $\mathscr{H}(p^\infty)$ where p^∞ sends each p-group into its maximal divisible subgroup is related to a concept studied by S. A. Khabbaz. He calls a p-group *starred* [2] if it has the same power as a basic subgroup. Khabbaz and J. M. Irwin call a p-group *fully starred* if every subgroup is starred. The class $\mathscr{H}(p^\infty)$ is the class of fully starred p-groups.

LEMMA 2.3. *A reduced p-group A is starred if and only if A/C divisible implies* $|C| = |A|$.

PROOF. Suppose A is a reduced starred p-group, $C \subseteq A$ and A/C is divisible. If $C \neq A$, then C is infinite. It is therefore contained in a pure subgroup H of the same power [1, p. 78]. Then A/H is also divisible. Let B be a basic subgroup of H. Then B is pure in A and, as is easily shown, A/B is divisible. Hence B is a basic subgroup of A. Since A is starred, $|A| = |B| = |H| = |C|$.

The proof in the opposite direction is immediate since A/C is divisible when C is a basic subgroup of A.

THEOREM 2.4. $\mathscr{H}(p^\infty)$ *is the class of fully starred p-groups.*

PROOF. Suppose $A \in \mathscr{H}(p^\infty)$ and $C \subseteq A$. If C is finite, it is clearly starred, hence we may assume C infinite. By Theorem 2.2 (i) $C \in \mathscr{H}(p^\infty)$. Let B be a basic subgroup of C. Then B is also infinite and C/B is divisible. Finally let $B \subseteq D$ with $|B| = |D|$ and $p^\infty(C/D) = 0$. Then $D = C$ and $|C| = |B|$ showing that C is starred.

Suppose on the other hand that A is fully starred. Since a nonzero divisible group is not starred, A is reduced, i.e., $p^\infty A = 0$. Let B be an infinite subgroup of A and let $C \subseteq A$ be such that $C/B = p^\infty(A/B)$. Then $p^\infty(A/C) = 0$ and, since C is starred, $|B| = |C|$ by Lemma 2.3. Hence $A \in \mathscr{H}(p^\infty)$.

COROLLARY 2.5. *If C is a p-group which is not fully starred, then C is contained in no group of the form* Tor (A, B) *with A and B both reduced. In particular an uncountable p-group with a countable basic subgroup is contained in no such group.*

PROOF. According to Theorem 2.1 and Theorem 2.4 Tor (A, B) is fully starred if A and B are reduced p-groups.

3. This section continues the study of generalized purity begun in [5]. For convenience I shall repeat the pertinent definition from that paper. An extension

$$\mathfrak{G}\colon\ 0 \to Z \to G \to H \to 0$$

where Z is the group of integers gives rise to a subfunctor $S(\mathfrak{G})$ of the identity by

$$S(\mathfrak{G})A = \text{Image}\,(\text{Hom}\,(G, A) \to A)\ .$$

A subfunctor of the identity S is called a *cotorsion functor* if $S = S(\mathfrak{G})$ for some $\mathfrak{G} \in \text{Ext}\,(H, Z)$ with H a torsion group. An extension

$$0 \to C \to E \to A \to 0$$

is called *S-pure* if it belongs to $S\,\text{Ext}\,(A, C)$ and the group A is *S-projective* if $S\,\text{Ext}\,(A, C) = 0$ for every C (in [5] I used the more accurate term $S\,\text{Ext}$-projective). $S\,\text{Ext}$ is said to have enough projectives if, for each group A, there is an S-pure exact sequence

$$0 \to C \to E \to A \to 0$$

in which E is S-projective. According to Theorem 4.8 of [5] the functor $S\,\text{Ext}$ has enough projectives if and only if $S = S(\mathfrak{G})$ for some $\mathfrak{G} \in \text{Ext}\,(H, Z)$ where H is S-projective.

It has been shown that, for each torsion group H, there is at most one functor $S(\mathfrak{G})$ with $\mathfrak{G} \in \text{Ext}\,(H, Z)$ such that H is $S(\mathfrak{G})$-projective.

THEOREM 3.1. *If H is a reduced p-group which is contained in no group of the form* Tor (A, B) *with A and B reduced, then there is no $\mathfrak{G} \in$* Ext (H, Z) *such that H is $S(\mathfrak{G})$-projective.*

PROOF. If $\mathfrak{G} \in \text{Ext}\,(H, Z)$ and M is $S(\mathfrak{G})$-projective, then a torsion group C is $S(\mathfrak{G})$-projective if and only if C is a direct summand of Tor (H, C). This useful observation was omitted from [5] but it follows easily from the discussion in sections 3 and 4 of that paper. Now if H is a reduced p-group and is to be $S(\mathfrak{G})$-projective, then H is a direct summand of Tor (H, H) contrary to the properties assumed for H. The existence of groups H with the properties required in this theorem follows from Corollary 2.5.

The theory of cotorsion functors was developed to provide a suitable context in which to discuss the functors p^{α}. For each ordinal α, there is a group H_{α} and $\mathfrak{G}_{\alpha} \in \text{Ext}\,(H_{\alpha}, Z)$ such that

(i) $p^{\alpha} = S(\mathfrak{G}_{\alpha})$,

(ii) $H_{\alpha} = \sum_{\beta<\alpha} H_{\beta}$ if α is a limit ordinal,

(iii) $p^{\alpha}H_{\alpha+1}$ is cyclic of order p and $H_{\alpha+1}/p^{\alpha}\,H_{\alpha+1} \cong H_{\alpha}$,

(iv) $p^{\alpha}H_{\alpha} = 0$, and

(v) H_{α} is p^{α}-projective.

THEOREM 3.2. *A countable p-group A is p^{α}-projective if and only if $p^{\alpha}\,A = 0$.*

PROOF. If A is p^{α}-projective, then A is a direct summand of Tor (H_{α}, A). By Lemma 1.2 we then have $p^{\alpha}\,A = 0$.

If $\beta < \alpha$ and A is p^{β}-projective, then A is p^{α}-projective. Since A is countable, $p^{\alpha}A = 0$ implies $p^{\beta} A = 0$ for some countable ordinal β. Hence we need only prove the theorem for α countable.

Let A be a countable group and let $p^{\alpha} A = 0$ for α countable. Since H_{α} is countable when α is countable and $\lambda(H_{\alpha}) = \alpha$, we have $\lambda(A) \leq \lambda(H_{\alpha})$. Then Corollary 1.8 tells us that A is isomorphic to a direct summand of Tor (H_{α}, A) and is therefore p^{α}-projective.

The functor p^{α} Ext is called *hereditary* if it satisfies the following three equivalent conditions:

1) For each p^{α}-pure exact sequence

$$0 \to C \to E \to A \to 0$$

and each group B, the sequences

$$p^{\alpha}\, \mathrm{Ext}\,(E, B) \to p^{\alpha}\, \mathrm{Ext}\,(C, B) \to 0$$

and

$$p^{\alpha}\, \mathrm{Ext}\,(B, E) \to p^{\alpha}\, \mathrm{Ext}\,(B, A) \to 0$$

are exact.

2) Every p^{α}-pure subgroup of a p^{α}-projective group is p^{α}-projective.

3) For each group A, there is a p^{α}-pure exact sequence

$$0 \to M \to P \to A \to 0$$

in which both M and P are p^{α}-projective. The equivalence of these three conditions is well known in homology theory and is easily proved using the theory developed by the author [5] by imitating the proof of Proposition 3.7 of MacLane [4 p. 75].

THEOREM 3.3. *If α is a countable ordinal, then p^{α} Ext is hereditary.*

PROOF. Let α be a countable ordinal and let

$$\mathfrak{G}\colon\ 0 \to Z \to G \to H \to 0$$

be an exact sequence such that $H = H_{\alpha}$ and $p^{\alpha} = S(\mathfrak{G})$. Given A, Theorem 3.1 (iii) of [5] gives a p^{α}-pure exact sequence

$$0 \to M \to F \oplus \mathrm{Tor}\,(H, A) \xrightarrow{\varphi} A \to 0$$

where F is free and the restriction of φ to Tor (H, A) is the connecting homomorphism induced by $\mathfrak{G}$. Since F is free, the sequence

$$0 \to \mathrm{Tor}\,(G, A) \to \mathrm{Tor}\,(H, A) \to A$$

is exact, and Tor (H, A) is a torsion group, it is easy to verify that $M \cong N \oplus \mathrm{Tor}\,(G, A)$ where N is free.

Let T be the torsion subgroup of G. Then Tor $(G, A) =$ Tor (T, A). Moreover $T \cap Z = 0$ so that T is isomorphic to a subgroup of H. If α is countable, then H is also countable and $p^{\alpha} H = 0$. Hence T is countable with $p^{\alpha} T = 0$ so that T is p^{α}-projective by Theorem 3.2. Then Theorems 4.1 and 4.2 of [5] show that $F \oplus \operatorname{Tor}(H, A)$ and $N \oplus \operatorname{Tor}(T, A)$ are p^{α}-projective. Thus to each group A we have associated an exact sequence meeting the condition (3) and have shown that p^{α} Ext is hereditary.

Theorem 4.2 of [5] says that if A is p^{α}-projective so is $\operatorname{Tor}(A, B)$. The next theorem gives a partial converse.

THEOREM 3.4. *If A and B are p-groups* $\operatorname{Tor}(A, B)$ *p^{α}-projective and $p^{\alpha} A \neq 0$, then B is p^{α}-projective.*

PROOF. If $p_{\alpha} A \neq 0$, there is an element a in $p^{\alpha} A$ with order p^n $(n > 0)$. Let Za be the subgroup of A generated by a and let F be a free group with one generator x. There is an exact sequence

$$0 \to Z \to F \to Za \to 0$$

in which 1 in Z maps onto $p^n x$ and x maps onto a. Since the inclusion $Za \to A$ is a monomorphism and the functor Ext is hereditary, there is a commutative diagram

$$\begin{array}{rccccccccc} & 0 & \to & Z & \to & F & \to & Za & \to & 0 \\ & & & \| & & \downarrow & & \downarrow & & \\ \mathfrak{G}\colon & 0 & \to & Z & \to & G & \to & A & \to & 0 \end{array}$$

in which the rows are exact [4, Proposition 3.7]. Since the map $F \to G$ is a monomorphism, F may be identified with its image in G. In the same way Z may be identified with its image in G. To summarize there is an element x in G mapping onto a modulo Z with $p^n x = 1$.

The next step is to show that x is in $p^{\alpha} G$. Suppose if possible that x is not in $p^{\beta} G$ for some $\beta \leq \alpha$ and let β be the least such ordinal. Since $p^{\beta} G = \bigcap_{\gamma < \beta} p^{\gamma} G$ when β is a limit ordinal, $\beta = \gamma + 1$ and x is in $p^{\beta} G$. Then $Z = p^n (Zx) \subseteq p^{\beta} G$. It is easy to show by induction on β that each element of G mapping into $p^{\alpha} A$ modulo Z lies in $p^{\beta} G$. Hence x is in $p^{\beta} G$ contrary to hypothesis.

Thus x is in $p^{\alpha} G$ and $1 = p^n x \in p^{\alpha+n} G$. It follows from the definition of $S(\mathfrak{G})$ that $S(\mathfrak{G}) C \subseteq p^{\alpha+n} C$ for all C.

By Theorem 3.1 of [5] there is, for each group C, an exact sequence

$$0 \to S(\mathfrak{G}) \operatorname{Ext}(B, C) \to \operatorname{Ext}(B, C) \xrightarrow{\delta^*} \operatorname{Ext}(\operatorname{Tor}(A, B), C)$$

where $\delta\colon \operatorname{Tor}(A, B) \to B$ is the connecting homomorphism induced by $\mathfrak{G}$. The map δ_* carries $p^{\alpha} \operatorname{Ext}(B, C)$ into $p^{\alpha} \operatorname{Ext}(\operatorname{Tor}(A, B), C)$ which is 0 because $\operatorname{Tor}(A, B)$ is assumed p^{α}-projective.

Hence

$$p^{\alpha} \operatorname{Ext}(B, C) \subseteq S(\mathfrak{G}) \operatorname{Ext}(B, C) \subseteq p^{\alpha+n} \operatorname{Ext}(B, C).$$

Since $n > 0$, p Ext (B, C) is therefore p-divisible. However B is a p-group so that Ext (B, C) has no p-divisible subgroups by Proposition 4.1 of [5]. Therefore p^{α} Ext $(B, C) = 0$. Since C is arbitrary, B is p^{α}-projective.

A p-group is p^{ω}-projective if and only if it is a direct sum of cyclic groups. Hence the special case

COROLLARY 3.5. *If A and B are p-groups with $p^{\omega} A \neq 0$ and if* Tor (A, B) *is a direct sum of cyclic groups, then B is a direct sum of cyclic groups.*

Kolettis [3] showed that Ulm's Theorem holds for reduced groups which are direct sums of countable groups and raised the question whether every subgroup of such a group must be a direct sum of countable groups. The answer is no.

COROLLARY 3.6. *Let A be a reduced countable p-group with $p^{\omega} A \neq 0$ and let B be a p-group without elements of infinite height which is not a direct sum of cyclic groups. Then* Tor (A, B) *is contained in a direct sum of copies of A but is not a direct sum of countable groups.*

PROOF. As in the proof of Lemma 1.2 we have $B \subseteq \sum Z(p^{\infty})$ so that Tor $(A, B) \subseteq \sum$ Tor $(A, Z(p^{\infty})) \simeq \sum A$. Since $p^{\omega} B = 0$, p^{ω} Tor $(A, B) = 0$ by Lemma 1.2. A countable p-group without elements of infinite height is a direct sum of cyclic groups. Hence if Tor (A, B) is a direct sum of countable groups, it is a direct sum of cyclic groups. Now Corollary 3.5 and $p^{\omega} A \neq 0$ imply that B is a direct sum of cyclic groups contrary to hypothesis.

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ENDOMORPHISM RINGS OF PRIMARY ABELIAN GROUPS[1]

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Introduction

The work of Corner [1] has shown that the endomorphism rings of torsion-free Abelian groups can be quite general. In contrast, the endomorphism rings of primary Abelian groups have many special properties. In the author's work [3], a start was made in the study of the endomorphism rings of p-groups. It is the purpose of this paper to continue these investigations.

If G is a p-group without elements of infinite height, and B is a basic subgroup of G, then the endomorphism ring $\mathscr{E}(G)$ contains an embedded copy of the ring $\mathscr{E}_p(B)$ consisting of all bounded endomorphisms φ of G such that $\varphi(B) \subseteq B$. The subring $\mathscr{E}_p(B)$ has a number of special properties. Most important, it is a right ideal in $\mathscr{E}(G)$. In this paper, we will characterize up to isomorphism those extensions $\sum$ of a fixed ring $\mathscr{E}_p(B)$ (where B is an unbounded basic group) which are of the form $\mathscr{E}(G)$ for some group G without elements of infinite height which has B as its basic subgroup. At the same time the extensions of the form $\mathscr{E}_p(G)$ will also be characterized.

Our notation and terminology conform with those of the standard reference work of Fuchs on Abelian groups [2]. Similarly, we will use the standard results on Abelian groups without giving specific references to pages in [2]. In a few instances it will be necessary to use some special cases of theorems proved in Pierce [3]. These special theorems can be proved directly by elementary means—a task which we delegate to the reader.

1. p-Faithful rings

Let $\sum$ be an arbitrary ring. We will denote by $\sum_p$ the p-primary component of the torsion subgroup of $\sum$. That is, $\sum_p$ consists of all $\xi \in \sum$ such

[1] The main theorem of this paper was developed in the author's address at the Colloquium. One part of the address was devoted to a joint work of R. A. Beaumont and the author, which will be published elsewhere.

that $p^k \xi = 0$ for some $k \geq 0$. It is obvious that $\sum_p$ is an ideal of $\sum$. In fact, if $p^k \xi = 0$ and $\zeta \in \sum$, then by the distributive law $p^k(\xi\zeta) = p^k(\zeta\xi) = 0$.

1.1. Definition. A ring $\sum$ is called p-faithful if $\zeta \in \sum$ and $\zeta\xi = 0$ for all $\xi \in \sum_p$ implies $\zeta = 0$.

In other words, $\sum$ is p-faithful exactly when the left regular representation of $\sum$ on the ideal $\sum_p$ is faithful.

1.2. Lemma. *If $\sum$ is p-faithful, then $\bigcap_{n<\omega} p^n \sum = 0$.*

Proof. Let $\zeta \in \bigcap_{n<\omega} p^n \sum$. Suppose that $\xi \in \sum_p$, say $p^n \xi = 0$. Since $\zeta = p^n \eta$ for some $\eta \in \sum$, it follows that $\zeta\xi = p^n \eta\xi = 0$. Hence, $\zeta = 0$.

1.3. Corollary. *If $\sum$ is p-faithful, then the torsion subgroup of $\sum$ is p-primary without elements of infinite height.*

Proof. If $q \neq p$, then $p^n \sum_q = \sum_q$ for all n. Hence, $\sum_q = \bigcap_{n<\omega} p^n \sum_q \subseteq \bigcap_{n<\omega} p^n \sum = 0$. Hence, $\sum_p$ is the entire torsion subgroup of $\sum$. Since $\bigcap_{n<\omega} p^n \sum_p \subseteq \bigcap_{n<\omega} p^n \sum = 0$, it follows that $\sum_p$ has no elements of infinite height.

1.4. Lemma. *Let G be a reduced p-group. Suppose that $\sum$ is a subring of $\mathscr{E}(G)$ such that the subgroup H of G generated by $\{\xi(x) \mid \xi \in \sum_p, x \in G\}$ has the property that G/H is divisible. Then $\sum$ is p-faithful.*

Proof. Let $\zeta \in \sum$ be such that $\zeta\xi = 0$ for all $\xi \in \sum_p$. If K is the kernel of ζ, then $H \subseteq K$. Hence, $\zeta(G)$ is divisible. Since G is reduced, it follows that $\zeta(G) = 0$. Thus, $\zeta = 0$.

An interesting consequence of 1.3 and 1.4 is the fact that for any reduced p-group G, the ring $\mathscr{E}(G)$ has a faithful representation as a subring of a ring $\mathscr{E}(H)$, where H is a p-group without elements of infinite height.

2. Basic rings

We now consider a special situation. Let B be an unbounded basic p-group, that is, a direct sum of p-power cyclic groups. Collecting the cyclic summands of the same order, it is possible to write

$$B = \sum_{n<\omega} B_n,$$

where B_n is a direct sum of cyclic groups of order p^{n+1}. This decomposition is not canonical, but for our purposes it is convenient to select and fix such a decomposition once and for all.

Let $\bar{B}$ denote the closure (or torsion completion) of B. The group $\bar{B}$ can be considered as the torsion subgroup of the complete direct sum $\sum^*_{n<\omega} B_n$. It is of course assumed that B is imbedded as a pure subgroup in $\bar{B}$. If G is any p-group without elements of infinite height, which contains B as a basic subgroup, then it is well known that there is a B-isomorphism of G onto a pure subgroup of $\bar{B}$. Thus, in a sense the study of p-groups without elements of infinite height is equivalent to the study of the pure subgroups of suitable closed groups $\bar{B}$.

It is convenient to fix some notation which will be standard below.

2.1. NOTATION. Corresponding to the fixed decomposition

$$B = \sum_{n<\omega} B_n ,$$

let

$$\bar{B}_m = \{p^m \bar{B}, \sum_{n\geqq\omega} B_n\}.$$

Let π_m denote the projection of $\bar{B}$ on $B_0 \oplus B_1 \oplus \ldots \oplus B_{m-1}$ corresponding to the direct decomposition $\bar{B} = B_0 \oplus B_1 \oplus \ldots \oplus B_{m-1} \oplus \bar{B}_m$. Note that $\pi_0 = 0$. Define

$$\delta_m = \pi_{m+1} - \pi_m .$$

Then $\delta_m(\bar{B}) = B_m$, $\delta_m^2 = \delta_m$, and $\pi_m = \delta_0 + \delta_1 + \ldots + \delta_{m-1}$.

All of the endomorphisms π_m and δ_m have finite order. In fact, $p^m \pi_m = p^{m+1} \delta_m = 0$.

The following simple result will be needed later.

2.2. LEMMA. *If $x \in \bar{B}$, then* $E((1 - \pi_m)(x)) + h((1 - \pi_m)(x)) \geqq m + 1$.

PROOF. Clearly, $(1 - \pi_m)(x) \in \bar{B}_m$, and it is well known that for $w \in \bar{B}_m$ the inequality $E(w) + h(w) \geqq m + 1$ is satisfied.

By virtue of the p-adic topology in $\bar{B}$, it is possible to assign a meaning to certain infinite sums. Specifically, if $u_0, u_1, u_2, \ldots$ is a sequence of elements of $\bar{B}$ such that the exponents $E(u_n)$ are bounded and $\lim_{n\to\infty} h(u_n) = \infty$, then the partial sums $\sum_{n<m} u_n$ converge to an element of $\bar{B}$. It is convenient to denote this element by $\sum_{n<\omega} u_n$. As an example, the following useful identity holds for all $w \in \bar{B}$.

$$w = \sum_{n<\omega} \delta_n(w). \tag{2.3}$$

Suppose that $B \subseteq G \subseteq \bar{B}$, where G is a subgroup of $\bar{B}$. If φ is an endomorphism of G, then it is well known that φ has a unique extension to an endomorphism of $\bar{B}$. This extension can be defined by the identity

$$\bar{\varphi}(w) = \sum_{n<\omega} \varphi\delta_n(w).$$

Since $E(\varphi\delta_n(w)) \leqq E(w)$ and $h(\varphi\delta_n(w)) \geqq h(\delta_n(w)) \geqq n + 1 - E(w)$, this infinite sum is well defined. The fact that the extension of φ to $\bar{B}$ is unique follows easily from the divisibility of $\bar{B}/G$. Notice that the exponent of φ is preserved under the extension to $\bar{B}$.

The fact that endomorphisms have unique extensions to $\bar{B}$ under the conditions given above makes it possible to adopt a very useful convention. If $B \subseteq G \subseteq \bar{B}$, where G is a subgroup of $\bar{B}$, we will identify the endomorphism ring of G with the subring of the endomorphism ring of

$\bar{B}$ consisting of the extensions of the endomorphisms of G. That is,

$$\mathscr{E}(G) = \{\varphi \in \mathscr{E}(\bar{B}) \mid \varphi(G) \subseteq G\}.$$

This identification carries with it the identification of the torsion subgroup of $\mathscr{E}(G)$ (which we will denote by $\mathscr{E}_p(G)$) with a subring of $\mathscr{E}_p(\bar{B})$, namely

$$\mathscr{E}_p(G) = \{\varphi \in \mathscr{E}_p(\bar{B}) \mid \varphi(G) \subseteq G\}.$$

2.4. Lemma. *Suppose that $\varphi \in \mathscr{E}_p(\bar{B})$. Then $\varphi(\bar{B}) = \varphi(B)$.*

Proof. Let $E(\varphi) = k$. Since B is a basic subgroup of $\bar{B}$, we have $\bar{B} = B + p^k \bar{B}$. Hence, $\varphi(\bar{B}) = \varphi(B) + \varphi(p^k\bar{B}) = \varphi(B) + p^k\varphi(\bar{B}) = \varphi(B)$.

2.5. Corollary. *Let $\mathscr{E}_p(B) \subseteq \sum \subseteq \mathscr{E}(\bar{B})$, where $\sum$ is a subring of $\mathscr{E}(\bar{B})$. Then $\mathscr{E}_p(B)$ is a faithful right ideal of $\sum$.*

Proof. By 2.4, $\mathscr{E}_p(B)$ is a right ideal in $\sum$. Suppose that $\varphi\zeta = 0$ for all $\zeta \in \mathscr{E}_p(B)$. Then $\delta_n \zeta = 0$ for all n. Thus, by 2.3, $\zeta(w) = \sum_{n<\omega} \delta_n \zeta(w) = 0$ for all $w \in \bar{B}$. Hence, $\zeta = 0$.

3. The layer topology

Before further progress can be made, it is necessary to introduce a topology on rings of endomorphisms.

3.1 Definition. Let $\sum$ be a subring of $\mathscr{E}(\bar{B})$. For each non-negative integer k, define $N_k = \{\zeta \in \sum \mid \zeta(\bar{B}[p^k]) = 0\}$. The family of all such sets N_k constitutes a neighborhood basis at 0 for a topology on $\sum$ which we will call the *layer topology.*

The layer topology on rings of endomorphisms is metrizable. In fact, for ξ, η in $\sum$, define

$$d(\xi, \eta) = \inf\{p^{-n} \mid (\xi - \eta)(\bar{B}[p^n]) = 0\}.$$

It is easy to see that d is an invariant metric on $\sum$, and that

$$N_k = \{\zeta \in \sum \mid d(\zeta, 0) \leq p^{-k}\}.$$

Therefore, the topology determined by this metric is the layer topology.

3.2 Lemma. *Let $\sum$ be a subring of $\mathscr{E}(\bar{B})$, and suppose that $\zeta \in \sum$. Then $\zeta \in N_k$ if and only if $\zeta(B[p^k]) = 0$.*

Proof. Suppose that $\zeta(B[p^k]) = 0$. Let $x \in \bar{B}[p^k]$. For any $n \geq 0$, it is possible to write $x = b + p^n y$, where $b \in B[p^k]$ and $y \in \bar{B}$. Hence, $\zeta(x) = \zeta(b) + p^n \zeta(y) = p^n \zeta(y) \in p^n \bar{B}$. Since n can be arbitrary and $\bar{B}$ has no elements of infinite height, if follows that $\zeta(x) = 0$. This shows that $\zeta(B[p^k]) = 0$ implies $\zeta \in N_k$, and the opposite implication is trivial.

3.3 Corollary. *Let $B \subseteq G \subseteq \bar{B}$, where G is subgroup of $\bar{B}$. Then the layer topology on $\mathscr{E}(G)$ coincides with the topology obtained by taking all sets $\{\zeta \in \mathscr{E}(G) \mid \zeta(G[p^k]) = 0\}$, $k = 0, 1, 2, \ldots$, as a neighbourhood basis of 0.*

We now establish that multiplication is right and left continuous in the layer topology. It is convenient to use the metric defined above.

3.4 LEMMA. *Let ξ, η, and ζ belong to $\mathscr{E}(\bar{B})$. Then*

$$d(\xi\zeta, \eta\zeta) \leqq d(\xi, \eta) \text{ and } d(\zeta\xi, \zeta\eta) \leqq d(\xi, \eta)\,.$$

Hence, right and left multiplication are continuous in the layer topology.

PROOF. Let $d(\xi, \eta) = p^{-n}$. Then we have $(\xi - \eta)(\bar{B}[p^n]) = 0$. Hence, $(\zeta\xi - \zeta\eta)(\bar{B}[p^n]) = \zeta(\xi - \eta)(\bar{B}[p^n]) = 0$ and $(\xi\zeta - \eta\zeta)(\bar{B}[p^n]) = (\xi - \eta)\zeta(\bar{B}[p^n]) \subseteq (\xi - \eta)(\bar{B}[p^n]) = 0$. Thus, $d(\xi\zeta, \eta\zeta) \leqq p^{-n}$ and $d(\zeta\xi, \zeta\eta) \leqq \leqq p^{-n}$.

We will need one more simple result concerning the layer topology.

3.5 LEMMA. *Let $\zeta \in \mathscr{E}_p(\bar{B})$. Then $\lim_{n\to\infty} \zeta\pi_n = \zeta$ in the layer topology.*

PROOF. Let $p^k\zeta = 0$. Suppose that m is given. Let $n \geqq m + k - 1$. If $x \in G[p^m]$, then $E(x) \leqq m$. Hence, $E((1 - \pi_n)(x)) \leqq m$. It follows from 2.2 that $h((1 - \pi_n)(x)) \geqq k$. That is, $(1 - \pi_n)(x) = p^k y$ for some $y \in \bar{B}$. Therefore, $\zeta(1 - \pi_n)(x) = \zeta(p^k y) = p^k\zeta(y) = 0$. This shows that $(\zeta - \zeta\pi_n)(\bar{B}[p^m]) = 0$. Since m was arbitrary, $\lim_{n\to\infty} \zeta\pi_n = \zeta$ in the layer topology.

4. The first representation theorem

In Section 2. it was shown that if $\sum$ is a subring of $\mathscr{E}(\bar{B})$, and if $\mathscr{E}_p(B) \subseteq \subseteq \sum$, then $\mathscr{E}_p(B)$ is a faithful right ideal in $\sum$. Moreover, by 3.4 right multiplication by elements of $\sum$ is a continuous transformation on $\mathscr{E}_p(B)$ in the layer topology. We will now prove that these two conditions are characteristic.

4.1 THEOREM. *Let $\sum$ be a ring which contains $\mathscr{E}_p(B)$ as a faithful right ideal. Suppose that the mapping $\varphi \to \varphi\zeta$ is continuous in the layer topology of $\mathscr{E}_p(B)$ for all $\zeta \in \sum$. Then there is a ring isomorphism of $\sum$ into $\mathscr{E}(\bar{B})$ which is the identity on $\mathscr{E}_p(B)$.*

PROOF. For $\zeta \in \sum$ and $w \in \bar{B}$, define

$$\lambda_\zeta(w) = \sum\nolimits_{n<\omega}(\delta_n\,\zeta)(w).$$

Note that $(\delta_n\,\zeta)(w) \in B_n$, $E(\delta_n\,\zeta(w)) \leqq E(w)$, and $h(\delta_n\,\zeta(w)) \geqq n + 1 - E(w)$. Consequently, the right hand sum represents a well-defined element of $\bar{B}$. Clearly, λ_ζ is a homomorphism of $\bar{B}$ into $\bar{B}$. If $\zeta \in \mathscr{E}_p(B)$, then, $\lambda_\zeta(w) = = \sum_{n<\omega}(\delta_n\,\zeta)(w) = \sum_{n<\pi}\delta_n(\zeta(w)) = \zeta(w)$ for all $w \in \bar{B}$. Therefore, $\lambda_\zeta = \zeta$ for all $\zeta \in \mathscr{E}_p(B)$. From the distributive law in $\sum$, it follows that $\lambda_{\xi-\eta} = = \lambda_\xi - \lambda_\eta$. We wish to prove that $\lambda_{\xi\eta} = \lambda_\xi\,\lambda_\eta$. Let $m < \omega$ be arbitrary. Since $\delta_m\,\xi$ is in $\mathscr{E}_p(\bar{B})$, we have $\lim_{n\to\infty}\delta_m\,\xi\,\pi_n = \delta_m\,\xi$ in the layer topology by 3.5. That is, $\lim_{n\to\infty}\sum_{r<n}\delta_m\,\xi\,\delta_r = \delta_m\,\xi$. Therefore, since

right multiplication by elements of $\sum$ is continuous in the layer topology, it follows that $\lim_{n\to\infty} \sum_{r<n} \delta_m \xi \delta_r \eta = \delta_m \xi \eta$. Let $w \in \bar{B}$ and $E(w) = k$. Then an integer s exists such that $\sum_{r<n} \delta_m \xi \delta_r \eta(w) = \delta_m \xi \eta(w)$ for all $n \geq s$. Consequently, $\delta_m \xi \delta_r \eta(w) = 0$ if $r > s$, so that $\delta_m \lambda_\xi \lambda_\eta(w) = \delta_m \xi\left(\sum_{r<\omega} \delta_r \eta(w)\right) = \sum_{r<\omega} \delta_m \xi \delta_r \eta(w) = \sum_{r\leq s} \delta_m \xi \delta_r \eta(w) = \delta_m \xi\eta(w) = \delta_m \lambda_{\xi\eta}(w)$. Thus, by 2.3, $\lambda_\xi \lambda_\eta(w) = \lambda_{\xi\eta}(w)$. Since w is arbitrary, $\lambda_{\xi\eta} = \lambda_\xi \lambda_\eta$. To complete the proof, we show that if $\lambda_\zeta = 0$ for $\zeta \in \sum$, then $\zeta = 0$. If $\lambda_\zeta = 0$, then $\xi \in \mathscr{E}_p(B)$ implies $\xi \zeta = \lambda_{\xi\zeta} = \lambda_\xi \lambda_\zeta = 0$. Since $\mathscr{E}_p(B)$ is a faithful right ideal in $\sum$, it follows that $\zeta = 0$.

5. Purity

Our ultimate aim is to characterize the rings $\mathscr{E}(G)$, where $B \subseteq G \subseteq \bar{B}$, and G is a pure subgroup of $\bar{B}$. An intermediate goal is the characterization of the rings $\mathscr{E}_p(G)$. In this section, the purity of G in $\bar{B}$ is considered.

5.1 Lemma. *Let $B \subseteq G \subseteq \bar{B}$, where G is a pure subgroup of $\bar{B}$. Then $\mathscr{E}_p(G)$ is a pure subgroup of $\mathscr{E}_p(\bar{B})$.*

Proof. It follows from 2.4 that $\mathscr{E}_p(G)$ is a subgroup of $\mathscr{E}_p(\bar{B})$. Moreover, since G is pure in $\bar{B}$, it follows that $\mathrm{Hom}_s(B, G)$ is pure in $\mathrm{Hom}_s(B, \bar{B})$ [3, (5.10)]. The torsion subgroup of $\mathrm{Hom}(B, \bar{B})$ is contained in $\mathrm{Hom}_s(B, \bar{B})$, and by the convention on extending homomorphisms to $\bar{B}$, the torsion subgroup may be identified with $\mathscr{E}_p(\bar{B})$. Under this identification the torsion subgroup of $\mathrm{Hom}_s(B, G)$ goes into $\mathscr{E}_p(G)$. It follows easily that $\mathscr{E}_p(G)$ is pure in $\mathscr{E}_p(\bar{B})$.

This lemma points out the importance of characterizing the pure subrings of $\mathscr{E}_p(\bar{B})$.

5.2 Lemma. *Suppose that $\mathscr{E}_p(B) \subseteq \sum \subseteq \mathscr{E}_p(\bar{B})$, where $\sum$ is a subring of $\mathscr{E}_p(\bar{B})$. Then $\sum$ is pure in $\mathscr{E}_p(\bar{B})$ if and only if for any ζ in $\sum$, if $\mathscr{E}_p(B)\zeta \subseteq p^k \mathscr{E}_p(B)$, then $\zeta \in p^k \sum$.*

Proof. Suppose that $\sum$ is pure, $\zeta \in \sum$, and $\mathscr{E}_p(B)\zeta \subseteq p^k \mathscr{E}_p(B)$. If $n \geq 0$, then $\delta_n \zeta = p^k \varphi_n$ for some $\varphi_n \in \mathscr{E}_p(B)$. Thus, if $w \in \bar{B}[p^k]$, $\delta_n \zeta(w) = p^k \varphi_n(w) = \varphi_n(p^k w) = 0$ for all $n < \omega$. Hence, $\zeta(\bar{B}[p^k]) = 0$. Next, let $w \in \bar{B}$ be arbitrary. Suppose that $h(\zeta(w)) = s$. Then for some n, $h(\delta_n \zeta(w)) = s$. Since $\delta_n \in \mathscr{E}_p(B)$, it follows that $h(\delta_n \zeta(w)) \geq k$. Thus, $s \geq k$. This argument shows that $\zeta(\bar{B}) \subseteq p^k \bar{B}$. It follows from 5.9 [3] that $\zeta \in p^k \mathscr{E}_p(\bar{B})$, so that by the purity of $\sum$, $\zeta \in p^k \sum$. Conversely, suppose that for every $\zeta \in \sum$, $\mathscr{E}_p(B)\zeta \subseteq p^k \mathscr{E}_p(B)$ implies $\zeta \in p^k \sum$. Let $\zeta \in p^k \mathscr{E}_p(\bar{B}) \cap \sum$. Since $\mathscr{E}_p(B)$ is a right ideal in $\mathscr{E}_p(\bar{B})$, it follows that $\mathscr{E}_p(B)\zeta \subseteq p^k \mathscr{E}_p(B)$. Thus, $\zeta \in p^k \sum$. Consequently, $p^k \mathscr{E}_p(\bar{B}) \cap \sum = p^k \sum$, and $\sum$ is pure in $\mathscr{E}_p(\bar{B})$.

6. A Galois connection

The purpose of this section and the following one is to characterize the rings which are isomorphic to $\mathscr{E}_p(G)$ for some group G satisfying $B \subseteq G \subseteq \bar{B}$, G pure in $\bar{B}$. The first part of this characterization involves the construction of a group corresponding to each pure subring $\sum$ of $\mathscr{E}_p(\bar{B})$ such that $\sum \supseteq \mathscr{E}_p(B)$.

6.1 Definition. Let $\mathscr{E}_p(B) \subseteq \sum \subseteq \mathscr{E}_p(\bar{B})$, where $\sum$ is a pure subring of $\mathscr{E}_p(\bar{B})$. Define

$$\mathbf{G}(\textstyle\sum) = \{\varphi(b) \mid \varphi \in \textstyle\sum, b \in B\}.$$

6.2 Lemma. *Let $\mathscr{E}_p(B) \subseteq \sum \subseteq \mathscr{E}_p(\bar{B})$, where $\sum$ is a pure subring of $\mathscr{E}_p(\bar{B})$. Suppose that $x \in \mathbf{G}(\sum)$ and $b \in B$ are such that*

(a) $E(b) \geq E(x)$, *and*

(b) $\bar{B} = \langle b \rangle \oplus K$.

Then there exists $\psi \in \sum$ such that $\psi(b) = x$ and $\psi(K) = 0$.

Proof. Since $x \in \mathbf{G}(\sum)$, there exists $\varphi \in \sum$ and $c \in B$ such that $x = \varphi(c)$. Let $\bar{B} = \langle d \rangle \oplus H$, where $d \in B$ and $E(d) \geq E(c)$. Such a decomposition exists because B is unbounded. Define $\lambda\colon \bar{B} \to B$ by $\lambda(d) = c$, $\lambda(H) = 0$. Then $\lambda \in \mathscr{E}_p(B) \subseteq \sum$. Let $\chi = \varphi \lambda \in \sum$. Assume that $E(x) = m$, $E(d) = m + k$, where $k \geq 0$, since $\chi(d) = x$. Then $\bar{B}[p^k] = \langle p^m d \rangle \oplus H[p^k] \subseteq \ker \chi$. Write $x = e + p^k y$, where $e \in B$ and $E(e) \leq E(x) = m$. Define $\delta \in \mathscr{E}_p(\bar{B})$ by $\delta(d) = e$, $\delta(H) = 0$. Then $\delta \in \mathscr{E}_p(B) \subseteq \sum$ and $\delta(p^m d) = p^m e = 0$. Consequently, $\bar{B}[p^k] \subseteq \ker \delta$. Consider $\chi - \delta$. We have $\chi - \delta \in \sum$ and $\chi - \delta = p^k \mu$, where $\mu \in \mathscr{E}_p(\bar{B})$ is defined by the conditions $\mu(H) = 0$, $\mu(d) = y$. Note that $p^{m+k} y = p^m(e + p^k y) = p^m x = 0$, so that $E(y) \leq m + k = E(d)$. Since $\sum$ is pure in $\mathscr{E}_p(\bar{B})$, there exists $\tau \in \sum$ such that $\chi - \delta = p^k \tau$. Then $\tau(p^k d) = p^k \tau(d) = \chi(d) - \delta(d) = x - e = p^k y$. Clearly, $E(p^k d) = m = E(x) \leq E(b)$. Consequently, there exists $\zeta \in \mathscr{E}_p(B) \subseteq \sum$ and $\eta \in \mathscr{E}_p(B) \subseteq \sum$ satisfying

$$\zeta(b) = p^k d, \ \zeta(K) = 0,$$

$$\eta(b) = e, \ \eta(K) = 0.$$

Let $\psi = \eta + \tau \zeta$. Then $\psi \in \sum$, $\psi(K) = 0$, and $\psi(b) = \eta(b) + \tau(\zeta(b)) = e + \tau(p^k d) = e + p^k y = x$.

6.3 Lemma. *Let $\mathscr{E}_p(B) \subseteq \sum \subseteq \mathscr{E}_p(\bar{B})$, where $\sum$ is pure in $\mathscr{E}_p(\bar{B})$. Then $\mathbf{G}(\sum)$ is a pure subgroup of $\bar{B}$ with $B \subseteq \mathbf{G}(\sum)$.*

Proof. Let $z, w \in \mathbf{G}(\sum)$. Choose $b \in B$ such that $E(b) \geq E(z)$, $E(b) \geq E(w)$, and $\langle b \rangle$ is a direct summand of $\bar{B}$. By 6.2, there exist ψ_1 and ψ_2 in $\sum$ such that $\psi_1(b) = z$ and $\psi_2(b) = w$. Consequently, $z - w = (\psi_1 - \psi_2)(b) \in \mathbf{G}(\sum)$. Therefore, $\mathbf{G}(\sum)$ is a subgroup of $\bar{B}$. If $\langle c \rangle$ is a direct summand of B, then $\langle c \rangle$ is a pure, finite subgroup of $\bar{B}$. Therefore, $\langle c \rangle$ is also a direct summand of $\bar{B}$. The projection ϱ of $\bar{B}$ onto $\langle c \rangle$ belongs to $\mathscr{E}_p(B) \subseteq \sum$. Hence, $\langle c \rangle \subseteq \mathbf{G}(\sum)$. Since B is a direct sum of cyclic groups, it follows that $B \subseteq \mathbf{G}(\sum)$. To prove that $\mathbf{G}(\sum)$ is pure in $\bar{B}$, suppose that $x \in \bar{B}$, and $y = p^k x \in \mathbf{G}(\sum)$. Choose $b \in B$ such that $E(b) \geq E(x)$ and $\bar{B} = \langle b \rangle \oplus K$. By 6.2, there exists $\psi \in \sum$ such that $\psi(b) = p^k x$, $\psi(K) = 0$. Define $\varphi \in \mathscr{E}_p(\bar{B})$ by the conditions $\varphi(b) = x$, $\varphi(K) = 0$. Then $\psi = p^k \varphi$. Thus, $\psi \in p^k \mathscr{E}_p(\bar{B}) \cap \cap \sum = p^k \sum$. Consequently, $\psi = p^k \chi$ for some $\chi \in \sum$. Moreover, $y = = p^k x = \psi(b) = p^k \chi(b) \in p^k\, \mathbf{G}(\sum)$. Hence, $\mathbf{G}(\sum)$ is pure in $\bar{B}$.

6.4 Lemma. *Suppose that $\mathscr{E}_p(B) \subseteq \sum \subseteq T \subseteq \mathscr{E}_p(\bar{B})$, where $\sum$ and T are pure subrings of $\mathscr{E}_p(\bar{B})$. Then $\mathbf{G}(\sum) \subseteq \mathbf{G}(T)$. Moreover, $\mathscr{E}_p(\mathbf{G}(\sum)) \subseteq \sum$. Finally, if G is a pure subgroup of $\bar{B}$ such that $B \subseteq G \subseteq \bar{B}$, then $\mathbf{G}(\mathscr{E}_p(G)) = = G$.*

Proof. Obviously $\mathbf{G}(\sum) \subseteq \mathbf{G}(T)$. If $\varphi \in \sum$ and $b \in B$, then $\varphi(b) \in \mathbf{G}(\sum)$ by the definition 6.1. Thus, by 2.4, $\varphi(\bar{B}) \subseteq \mathbf{G}(\sum)$, so that $\varphi \in \mathscr{E}_p(\mathbf{G}(\sum))$. This proves that $\sum \subseteq \mathscr{E}_p(\mathbf{G}(\sum))$. Assume that $B \subseteq G \subseteq \bar{B}$, where G is pure in $\bar{B}$. If $b \in B$ and $\varphi \in \mathscr{E}_p(G)$, then $\varphi(b) \in G$. Hence, $\mathbf{G}(\mathscr{E}_p(G)) \subseteq G$. On the other hand, suppose that $x \in G$. Let $b \in B$ be such that $E(b) \geq E(x)$ and $\bar{B} = \langle b \rangle \oplus K$. Define $\varphi \in \mathscr{E}_p(\bar{B})$ by $\varphi(b) = x$, $\varphi(K) = 0$.Then $\varphi \in \mathscr{E}_p(G)$, so that $x = \varphi(b) \in \mathbf{G}(\mathscr{E}_p(G))$. Therefore, $G = \mathbf{G}(\mathscr{E}_p(G))$.

7. The finite topology

If $\mathscr{E}_p(B) \subseteq \sum \subseteq \mathscr{E}_p(\bar{B})$, with $\sum$ pure in $\mathscr{E}_p(\bar{B})$, then in general the inclusion $\sum \subseteq \mathscr{E}_p(\mathbf{G}(\sum))$ is proper. Our objective in this section is to obtain a topological criterion on $\sum$ which is necessary and sufficient for this inclusion to be equality.

7.1 Definition. Let $\sum$ be a ring. An idempotent $\pi \in \sum$ is called decomposable if it is possible to write $\pi = \pi_1 + \pi_2$, where π_1 and π_2 are non-zero, orthogonal idempotents ($\pi_1\pi_2 = \pi_2\pi_1 = 0$). An idempotent π is called indecomposable if $\pi \neq 0$ and π is not decomposable. An idempotent π in $\sum$ will be called *finite* if it is possible to write $\pi = \pi_1 + \pi_2 + \ldots + \pi_r$, where each π_i is an indecomposable idempotent and $\pi_i \pi_j = 0$ for $i \neq j$. The set of all finite idempotents in $\sum$ will be denoted by $\Phi(\sum)$.

In general, a ring $\sum$ may not have any indecomposable idempotents, in which case $\Phi(\sum)$ is empty. However, the rings we are concerned with do have an adequate supply of indecomposable idempotents.

7.2 Lemma. *Let $\mathscr{E}_p(B) \subseteq \sum \subseteq \mathscr{E}(\bar{B})$, where $\sum_p$ is a pure subring of $\mathscr{E}_p(\bar{B})$. Suppose that $0 \neq c \in \mathbf{G}(\sum_p)$ is such that $\bar{B} = \langle c \rangle \oplus D$. Let π be the projection of $\bar{B}$ onto $\langle c \rangle$ which is determined by this decomposition. Then π is an indecomposable idempotent in $\sum_p$. Conversely, every indecomposable idempotent in $\sum_p$ is of this form.*

PROOF. 1) Let $E(c) = m$. We will first show that $\bar{B} = \langle \pi_{2m}(c)\rangle \oplus \oplus D$ and $E(\pi_{2m}(c)) = m$. By 2.2, $h(c - \pi_{2m}(c)) > 2m - E(c - \pi_{2m}(c)) \geq m$. Consequently, if $c - \pi_{2m}(c) = j \cdot c + d$ with $d \in D$ and j an integer, then since the decomposition $\bar{B} = \langle c\rangle \oplus D$ is direct, we have $m < h(c - \pi_{2m}(c)) = \min \{h(j \cdot c), h(d)\} \leq h(jc)$. Since $\langle c\rangle$ is a direct summand of $\bar{B}$, it follows that if $jc \neq 0$, then $h(jc) < m$. Therefore, $jc = 0$ and $c - \pi_{2m}\, c \in D$. Consequently, $\langle \pi_{2m}\, c\rangle + D = \bar{B}$. Moreover, $D[p] \subset \bar{B}[p] = \langle p^{m-1}c\rangle + D[p] = = \langle p^{m-1}\pi_{2m}(c)\rangle + D[p]$, so that $p^{m-1}\pi_{2m}(c) \notin D[p]$, and in particular $p^{m-1}\pi_{2m}(c) \neq 0$. Therefore, $E(\pi_{2m}(c)) = m$ and $\langle \pi_{2m}(c)\rangle \cap D = 0$. Hence, $\bar{B} = \langle \pi_{2m}(c)\rangle \oplus D$.

2) We will now show that the projection π of $\bar{B}$ onto $\langle c\rangle$ corresponding to the decomposition $\bar{B} = \langle c\rangle \oplus D$ is in $\sum$. It is obvious that π is an indecomposable idempotent. By 1), $\bar{B} = \langle \pi_{2m}(c)\rangle \oplus D$, where $E(\pi_{2m}(c)) = = m = E(c)$. By 6.2, there exists $\psi \in \sum_p$ such that $\psi(\pi_{2m}(c)) = c$. Let ϱ be the mapping of $\bar{B}$ onto B determined by the conditions $\varrho(D) = 0$, $\varrho(c) = \pi_{2m}(c)$. This ϱ is well defined since $\bar{B} = \langle c\rangle \oplus D$ and $E(c) = = E(\pi_{2m}(c))$. Note that $\varrho \in \mathscr{E}_p(B) \subseteq \sum$. The product $\psi\varrho$ is in $\sum$ and $\psi\varrho$ satisfies $\psi\,\varrho\,(D) = 0$, $\psi\,\varrho(c) = \psi(\pi_{2m}(c)) = c$. Therefore, $\psi\,\varrho = \pi$, and we have the desired result that $\pi \in \sum$. Obviously then $\pi \in \sum_p$.

3) Suppose now that π_1 is any indecomposable idempotent in $\sum_p$. By 2.4, $\pi_1(\bar{B}) = \pi_1(B) \subseteq \mathbf{G}(\sum_p)$. Since $\pi_1 \neq 0$, there is a non-zero $c \in \pi_1(\bar{B}) \subseteq \subseteq \mathbf{G}(\sum_p)$ such that $\pi_1(\bar{B}) = \langle c\rangle \oplus E$. Thus, $\bar{B} = \langle c\rangle \oplus D$, where $D = = E \oplus (1 - \pi_1)(\bar{B})$. To complete the proof, we have to show that $E = 0$. Let π be the projection of $\bar{B}$ on $\langle c\rangle$ with the kernel D. Then $(\pi - \pi\pi_1)(\bar{B}) = = \pi(1-\pi_1)(\bar{B}) \subset \pi(D) = 0$ and $(\pi - \pi_1\pi)(\bar{B}) = (1-\pi_1)\pi(\bar{B}) = (1-\pi_1)(\langle c\rangle) = = 0$. Hence $\pi_1 = \pi + (\pi_1 - \pi)$, where $(\pi_1 - \pi)^2 = \pi_1 - \pi$, and $\pi(\pi_1 - \pi) = (\pi_1 - \pi)\pi = 0$. Since π_1 is indecomposable and $\pi \neq 0$, it follows that $\pi_1 = \pi$.

7.3 COROLLARY. *Let $\mathscr{E}_p(B) \subseteq \sum \subseteq \mathscr{E}(\bar{B})$, where $\sum_p$ is a pure subring of $\mathscr{E}_p(\bar{B})$. If $\pi \in \Phi(\sum_p)$, then $\pi(\bar{B})$ is a finite subgroup of $\mathbf{G}(\sum_p)$. Conversely, every finite subset of $\mathbf{G}(\sum_p)$ is contained in $\pi(\bar{B})$ for some $\pi \in \Phi(\sum_p)$.*

Indeed, it is clear from 7.2 and 6.3 that every finite direct summand of $\mathbf{G}(\sum_p)$ is of the form $\pi(\bar{B})$ for some $\pi \in \Phi(\sum_p)$, and since $\bar{B}$ has no elements of infinite height, every finite subset of $\mathbf{G}(\sum_p)$ can be embedded in a finite direct summand.

On the basis of 7.3 it is possible to give an abstract definition of the finite topology for the class of rings which we are studying.

7.4 DEFINITION. Let $\mathscr{E}_p(B) \subseteq \sum \subseteq \mathscr{E}(\bar{B})$, where $\sum_p$ is a pure subring of $\mathscr{E}_p(\bar{B})$. For each $\pi \in \Phi(\sum^p)$, define

$$N_\pi = \{\zeta \in \textstyle\sum \mid \zeta\,\pi = 0\}\,.$$

The family $\{N_\pi \mid \pi \in \Phi(\sum_p)\}$ constitutes a neighborhood basis at 0 for a topology on $\sum$ which we will call the finite topology.

In order to justify this definition, note that $\zeta \in N_\pi$ if and only if $\pi(\bar{B}) \subseteq \ker \zeta$. Thus, it follows from 7.3 that if π_1 and π_2 belong to $\Phi(\sum_p)$, then there exists $\pi_3 \in \Phi(\sum_p)$ such that $N_{\pi_1} \cap N_{\pi_2} \supseteq N_{\pi_3}$. Moreover, if $\zeta \in N_\pi$ for all $\pi \in \Phi(\sum_p)$, then $\zeta(B) = 0$. Consequently, $\zeta = 0$ since $\bar{B}$ is reduced and $\bar{B}/B$ is divisible. Hence, the finite topology satisfies the T_1 separation axiom.

If $\sum = \mathscr{E}(G)$ for a pure subgroup G of $\bar{B}$ with $B \subseteq G$, then the finite topology, as defined in 7.4, agrees with the usual finite topology for transformation groups. In fact, by 7.3 and 6.4, if K is a subset of $\mathscr{E}(G)$ and $\zeta \in \mathscr{E}(G)$, then ζ belongs to the closure of K in the topology of 7.4 if and only if, for every $F \subseteq G = \mathbf{G}(\mathscr{E}_p(G))$, there exists $\eta \in K$ such that $(\zeta - \eta)(F) = 0$. However, this is just the usual definition of the closure in the finite topology of transformation groups.

7.5 LEMMA. *Let $B \subseteq G \subseteq \bar{B}$, where G is a pure subgroup of $\bar{B}$. Then $\mathscr{E}_p(G)$ is dense in $\mathscr{E}(G)$ in the finite topology.*

PROOF. Let $\varphi \in \mathscr{E}(G)$. Let $\pi \in \Phi(\mathscr{E}_p(G))$. Then $\varphi\pi \in \mathscr{E}_p(\bar{B})$. Hence, by 2.4, 6.4, and 7.3, $\varphi\pi(\bar{B}) = \varphi\pi(B) \subseteq \varphi(\mathbf{G}(\mathscr{E}_p(G))) = \varphi(G) \subseteq G$. Hence, $\varphi\pi \in \mathscr{E}_p(G)$. Clearly, $\varphi - \varphi\pi \in N_\pi$.

In order to characterize the rings $\mathscr{E}(G)$ and $\mathscr{E}_p(G)$, it is necessary to use the concepts of completeness and torsion completeness relative to the finite topology. The definitions are essentially standard, but for convenience we will review them.

Let $\mathscr{E}_p(B) \subseteq \sum \subseteq \mathscr{E}(\bar{B})$, where $\sum_p$ is a pure subring of $\mathscr{E}_p(\bar{B})$. A subset of $\sum$ which is indexed by a directed set is called a net in $\sum$. We will generally use the notation $\{\zeta_i \mid i \in D\}$ to denote a net in $\sum$.

A net $\{\zeta_i \mid i \in D\}$ is said to converge to an element $\zeta \in \sum$ in the finite topology of $\sum$ if for each $\pi \in \Phi(\sum_p)$, there is an index $j \in D$ such that $\zeta_i - \zeta \in N_\pi$ for all $i \geqq j$ (that is, $\zeta_i\pi = \zeta\pi$ for $i \geqq j$). In this case we will write $\zeta = \lim_{i \in D} \zeta_i$. A net $\{\zeta_i \mid i \in D\}$ is called a Cauchy net if, for each $\pi \in \Phi(\sum_p)$, there is an index $j \in D$ such that $\zeta_i - \zeta_{i'} \in N_\pi$ for all i and i' which are $\geqq j$.

It is obvious that every convergent net is a Cauchy net. If conversely every Cauchy net converges to some $\zeta \in \sum$, then $\sum$ is said to be complete in the finite topology.

If $\{\zeta_i \mid i \in D\}$ is a net in $\sum$ such that there is some $k \geqq 0$ with the property that $E(\zeta_i) \leqq k$ for all i, then this net will be called bounded. If $\lim_{i \in D} \zeta_i = \zeta$ and $p^k \zeta_i = 0$ for all i, then $p^k \zeta = 0$. Therefore, a bounded net can only converge to a bounded element. The ring $\sum$ will be called torsion complete if $\sum_p = \sum$ and every bounded Cauchy net has a limit in $\sum$.

7.6 LEMMA. *Let $B \subseteq G \subseteq \bar{B}$, where G is a pure subgroup of $\bar{B}$. Then $\mathscr{E}(G)$ is complete and $\mathscr{E}_p(G)$ is torsion complete in their finite topologies.*

Proof. Let $\{\zeta_i \mid i \in D\}$ be a Cauchy net in $\varepsilon(G)$. If $x \in G$, there exists $j \in D$ such that $\zeta_i(x) = \zeta_j(x)$ for all $i \geqq j$. This follows from 7.3. Define $\zeta(x) = \zeta_j(x) \in G$. It is easy to verify that ζ defined in this way is a well defined endomorphism of G. Thus, ζ has a unique extension to $\bar{B}$. If $\pi \in \Phi\left(\mathscr{E}_p(G)\right)$, then since $\{\zeta_i \mid i \in D\}$ is Cauchy, there exists j such that $\zeta_i \pi = \zeta_j \pi$ for all $i \geqq j$. Consequently, $\zeta_i \pi = \zeta \pi$ for all $i \geqq j$. Hence, $\lim_{i \in D} \zeta_i = \zeta$. This proves that $\mathscr{E}(G)$ is complete in the finite topology. Suppose now that $\{\zeta_i \mid i \in D\}$ is a bounded Cauchy net in $\mathscr{E}_p(G)$, say $p^k \zeta_i = 0$ for all $i \in D$. Since $\{\zeta_i \mid i \in D\}$ is a Cauchy net in the finite topology of $\mathscr{E}_p(G)$, it is also a Cauchy net in the finite topology of $\mathscr{E}(G)$. In fact, it is obvious from the Definition 7.4 that the finite topology on $\mathscr{E}_p(G)$ is the same as the relative topology which it inherits as a subset of $\mathscr{E}(G)$ with the finite topology. Let $\zeta = \lim_{i \in D} \zeta_i$ in $\mathscr{E}(G)$. As we have just seen, this limit exists. Then $p^k \zeta = \lim_{i \in D} p^k \zeta_i = 0$. Hence, $\zeta \in \mathscr{E}_p(G)$ and $\lim_{i \in D} \zeta_i = \zeta$ in the finite topology of $\mathscr{E}_p(G)$. Consequently, $\mathscr{E}_p(G)$ is torsion complete.

We are now ready to prove the converse of the second half of 7.6.

7.7 Lemma. *Let $\mathscr{E}_p(B) \subseteq \sum \subseteq \mathscr{E}_p(\bar{B})$, where $\sum$ is a pure subring of $\mathscr{E}_p(\bar{B})$. Suppose that $\sum$ is torsion complete in its finite topology. Then $\sum = \mathscr{E}_p(\mathbf{G}(\sum))$.*

Proof. By 6.4, $\sum \subseteq \mathscr{E}_p(\mathbf{G}(\sum))$. Let $\zeta \in \mathscr{E}_p(\mathbf{G}(\sum))$. Let $\{F_i \mid i \in D\}$ be the set of all finite subsets of $\mathbf{G}(\sum)$, indexed by the directed set D in such a way that $i \leqq j$ if and only if $F_i \subseteq F_j$. We will construct a net $\{\zeta_i \mid i \in D\}$ in $\sum$ such that $(\zeta - \zeta_i)(F_i) = 0$ and $E(\zeta_i) \leqq E(\zeta)$ for all $i \in D$. It will then follow that $\{\zeta_i \mid i \in D\}$ is a bounded Cauchy net. This net has a limit $\eta \in \sum$, since $\sum$ is torsion complete. Clearly, $(\zeta - \eta)(B) = 0$, so that $\zeta = \eta \in \sum$ by 2.4. In order to construct ζ_i, let k be maximum exponent of the elements of F_i, so that $p^k f = 0$ for all $f \in F_i$. By 3.5, there is an n such that $(\zeta - \zeta\pi_n)(F_i) \subseteq (\zeta - \zeta\pi_n)\left(\bar{B}[p^k]\right) = 0$. The set $\pi_n F_i$ is contained in $B_0 \oplus B_1 \oplus \ldots B_{n-1} \subseteq B$, so that it is possible to write $B = \langle b_1 \rangle \oplus \oplus \langle b_2 \rangle \oplus \ldots \oplus \langle b_r \rangle \oplus C$, where $\pi_n F_i \subseteq \langle b_1 \rangle \oplus \langle b_2 \rangle \oplus \ldots \oplus \langle b_r \rangle$. This finite sum is pure in B, hence also in $\bar{B}$, so that it is a direct summand if $\bar{B}$. Let $\bar{B} = \langle b_1 \rangle \oplus \langle b_2 \rangle \oplus \ldots \oplus \langle b_r \rangle \oplus D$. By 6.2, there exist $\xi_i \in \sum$ such that $\xi_i(b_i) = \zeta(b_i)$, $\xi_i(b_j) = 0$ if $j \neq i$. Let $\xi = \xi_1 + \xi_2 + \ldots + \xi_r$. Then $\xi \in \sum$ and $\xi(b_i) = \zeta(b_i)$ for all i. Consequently, $(\xi - \zeta)(\pi_n F_i) = 0$. It is clear from the definition of ξ that $E(\xi) \leqq E(\zeta)$. Let $\zeta_i = \xi\pi_n$. Then $\zeta_i \in \sum$ since $\xi \in \sum$ and $\pi_n \in \mathscr{E}_p(B) \subseteq \sum$. Also $E(\zeta_i) \leqq E(\xi) \leqq E(\zeta)$. Finally $(\zeta - \zeta_i)(F_i) = (\zeta - \zeta\pi_n)(F_i) - (\xi\pi_n - \zeta\pi_n)(F_i) = 0$. The proof is therefore complete.

It is now possible to obtain the main result of this section.

7.8 Theorem. *Assume that B is an unbounded basic group. Let $\sum$ be a primary ring (that is, $\sum_p = \sum$) which contains $\mathscr{E}_p(B)$ as a right ideal. Assume that the following conditions hold:*

(a) *Right multiplication by elements of* $\sum$ *is a continuous transformation of* $\mathscr{E}_p(B)$ *in the layer topology.*

(b) *If* $\zeta \in \sum$ *is such that* $\mathscr{E}_p(B)\,\zeta \subseteq p^k\,\mathscr{E}_p(B)$, *then* $\zeta \in p^k \sum$.

(c) $\sum$ *is torsion complete in its finite topology.*

Then there is a group G *with* $B \subseteq G \subseteq \bar{B}$, G *pure in* $\bar{B}$, *and an isomorphism* λ *of* $\sum$ *onto* $\mathscr{E}_p(G)$ *such that* λ *is the identity mapping on* $\mathscr{E}_p(B)$.

Proof. Note that by condition (b), if $\mathscr{E}_p(B)\,\zeta = 0$, then $\zeta \in \bigcap_{k<\omega} p^k \sum = 0$ (see 1.2 and 1.4). Hence, $\mathscr{E}_p(B)$ is faithful. Consequently, by 4.1 and 5.2 there is an $\mathscr{E}_p(B)$-isomorphism λ of $\sum$ onto a pure subring of $\mathscr{E}(\bar{B})$. Since $\sum$ is primary, $\lambda(\sum) \subseteq \mathscr{E}_p(\bar{B})$. Moreover, $\lambda(\sum)$ is torsion complete. Thus, $\lambda(\sum) = \mathscr{E}_p(\mathbf{G}(\lambda(\sum)))$ by 7.7. Since $B \subseteq \mathbf{G}(\lambda(\sum)) \subseteq \bar{B}$ and $\mathbf{G}(\lambda(\sum))$ is pure in $\bar{B}$ by 6.3, the proof is complete.

8. The characterization of $\mathscr{E}(G)$

8.1 Theorem. *Assume that* B *is an unbounded basic group. Let* $\sum$ *be a ring which contains* $\mathscr{E}_p(B)$ *as a faithful right ideal. Assume that the following conditions hold:*

(a) *Right multiplication by elements of* $\sum$ *is a continuous transformation of* $\mathscr{E}_p(B)$ *in the layer topology.*

(b) *If* $\zeta \in \sum_p$ *is such that* $\mathscr{E}_p(B)\,\zeta \subseteq p^k\,\mathscr{E}_p(B)$, *then* $\zeta \in p^k \sum$.

(c) $\sum$ *is complete in its finite topology.*

Then there is a group G *with* $B \subseteq G \subseteq \bar{B}$, G *pure in* $\bar{B}$, *and an isomorphism* λ *of* $\sum$ *onto* $\mathscr{E}(G)$ *such that* λ *is the identity mapping on* $\mathscr{E}_p(B)$.

Proof. By 4.1, there is an $\mathscr{E}_p(B)$-isomorphism mapping $\sum$ into $\mathscr{E}(\bar{B})$. Note that $\lambda(\sum_p) = \lambda(\sum)_p = \mathscr{E}_p(\bar{B}) \cap \lambda(\sum)$. By 5.2 and the hypothesis (b), $\sum_p$ is pure in $\mathscr{E}_p(\bar{B})$. Since $\sum$ is complete in the finite topology, it follows easily that $\sum_p$ is torsion complete. Therefore, $\lambda(\sum_p)$ is also torsion complete. By 6.3 and 7.7, there is a group G with $B \subseteq G \subseteq \bar{B}$, G pure in $\bar{B}$, such that $\lambda(\sum_p) = \mathscr{E}_p(G)$. Suppose that $\zeta \in \lambda(\sum)$ and $x \in G$. Note that $G = \mathbf{G}(\mathscr{E}_p(G)) = \mathbf{G}(\lambda(\sum_p))$. Hence, by 7.3 there is a projection $\pi \in \lambda(\sum_p)$ such that $\pi(x) = x$. Since $\sum_p$ is an ideal in $\sum$, it follows that $\zeta\pi \in \lambda(\sum_p) = \mathscr{E}_p(G)$. Hence, $\zeta(x) = \zeta(\pi(x)) = \zeta\pi(x) \in G$. This shows that $\lambda(\sum) \subseteq \mathscr{E}(G)$. Assume now that $\zeta \in \mathscr{E}(G)$. By 7.5 $\mathscr{E}_p(G)$ is dense in $\mathscr{E}(G)$ in the finite topology. Hence, there is a net $\{\zeta_i \mid i \in D\} \subseteq \mathscr{E}_p(G) = \lambda(\sum_p)$ such that $\lim_{i \in D} \zeta_i = \zeta$ in the finite topology of $\mathscr{E}(G)$. That is, for any $\pi \in \Phi(\mathscr{E}_p(G)) = \Phi(\lambda(\sum_p)) = \Phi(\lambda(\sum)_p)$, there is a $j \in D$ such that $(\zeta_i - \zeta)\pi = 0$ for all $i \geq j$. It follows that $\{\zeta_i \mid i \in D\}$ is a Cauchy net in the finite topology of $\lambda(\sum)$. Since $\sum$ is complete, this sequence has a limit η in the finite topology of $\lambda(\sum)$. Evidently $(\zeta - \eta)\pi = 0$ for all $\pi \in \Phi(\mathscr{E}_p(G))$. Hence, $(\zeta - \eta)(G) = 0$, so that $\zeta = \eta \in \lambda(\sum)$. This completes the proof that $\lambda(\sum) = \mathscr{E}(G)$.

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FACTORIZATION OF CYCLIC GROUPS

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Introduction

A classical problem of Minkowski concerning the columnation of space-filling lattices was reduced by Hajós [3] to the following problem in finite Abelian groups. If such a group G is factorized directly as $G = A_1 + A_2 + \ldots + A_k$ where each A_i has the form $A_i = \{0, a_i, 2\,a_i, \ldots, (n_i-1)a_i\}$, is then one of the factors A_i a subgroup of G? Hajós showed that this is so and thus solved Minkowski's problem. The question was then posed as to whether all direct factorizations of finite Abelian groups can be obtained, no restriction being imposed on the form of the factors. This problem has proved very difficult and no general solution has been obtained, even in the case where two factors only are considered. In this case Hajós conjectured that one of the factors, A_1 say, must itself factorize as $A_1 = H + B_1$, where H is a non-zero subgroup of G. All factorizations into a sum of two factors of groups possessing this property have been found, but Hajós [4] and de Bruijn [1] soon showed that not all groups have this property. Hajós, de Bruijn, Rédei and Sands have classified all finite Abelian groups with respect to possessing this property. But the problem of obtaining the factorizations of those groups which do not possess this Hajós property remains. In this lecture we turn to the case of factorizations into sums of k factors, $k > 2$, but restrict our attention to finite cyclic groups.

All factorizations of these groups are obtained in which every factor has a prime power number of elements. So also are all factorizations in which every factor, except perhaps one, has an order, a power of a fixed prime. This leads to a complete solution of the problem for groups which are cyclic of order p^n or $p^n\,q$, where p and q are primes, and also for cyclic groups whose order has exponent sum[1] equal to the number of factors.

[1] If $N = p_1^{n_1} \ldots p_r^{n_r}$ the exponent sum $e(N)$ of N is $n_1 + \ldots + n_r$, where the p_i are distinct primes.

For groups not of order p^n or $p^n q$ the factorizations are constructed in which no factor has a non-zero subgroup as factor, where the exponent sum of the order of the group exceeds the number of factors by at least two. In the remaining case where the exponent sum of the order of the group exceeds the number of factors by one, all factorizations are found for $k = 3$ and for general k whenever the order of the group is $p^n q^{k+1-n}$, where p and q are distinct primes. But, in general, this case is left undecided.

Definitions and preliminary remarks

Throughout the talk group will mean finite cyclic additive group. If $A_1, A_2, \ldots, A_k$ are subsets of group G, each containing at least two elements, and if every element $g \in G$ may be expressed uniquely as $g = a_1 + a_2 + \ldots + a_k$, where $a_i \in A_i$, then $G = A_1 + A_2 + \ldots + A_k$ is called a k-factorization of G. A subset A of a group G is said to be periodic if there exists a non-zero subgroup H of G such that $A = H + B$. It may be assumed, by taking for H the set of all elements h such that $A + h = A$, that B is not periodic or consists of one element only. A group with the property that in every k-factorization at least one factor is periodic will be called a group with the Hajós k-property.

If G is the additive group of integers modulo N, and A is a subset of G consisting of the integers $\{a_1, a_2, \ldots, a_n\}$, then the polynomial $A(x)$ is defined to be $x^{a_1} + x^{a_2} + \ldots + x^{a_n}$. As $a \equiv b (\mathrm{mod}\ N)$ if and only if $x^a \equiv x^b\ (\mathrm{mod}\ (x^N - 1))$, the polynomial $A(x)$ is defined uniquely in the ring of polynomials modulo $(x^N - 1)$. With these definitions it follows that $G = A_1 + A_2 + \ldots + A_k$ if and only if

$$A_1(x) \cdot A_2(x) \ldots A_k(x) \equiv G(x) \equiv 1 + x + \ldots + x^{N-1} (\mathrm{mod}(x^N - 1)).$$

As $1 + x + \ldots + x^{N-1}$ is a factor of $x^N - 1$, it follows that each irreducible factor of $1 + x + \ldots + x^{N-1}$ will divide one of the polynomials $A_i(x)$. These irreducible factors are the cyclotomic polynomials whose roots are the d-th primitive roots of unity where $d \mid N$ and $d > 1$. We shall denote this polynomial by $F_d(x)$.

1. Some general results

LEMMA 1. *If H is a proper non-zero subgroup of a group G then there exists a non-periodic set C of coset representatives for G modulo H.*

PROOF. See de Bruijn [1, Lemma 1, case a].

LEMMA 2. *If A, B are non-periodic subsets of a group G and $A \subset H$ where H is a subgroup such that $H + B$ is a direct sum then $A + B$ is not periodic.*

PROOF. See Sands [11, Lemma].

LEMMA 3. *If G is a group possessing the Hajós k-property and G admits $(k+1)$-factorizations then G possesses the Hajós $(k+1)$-property, where $k > 2$.*

PROOF. Let $G = A_1 + \ldots + A_{k+1}$. Let $A_1 + A_2 = B$, $A_3 + A_4 = C$. Then $G = B + A_3 + \ldots + A_{k+1} = A_1 + A_2 + C + \ldots + A_{k+1}$. As G possesses the Hajós k-property it follows that in each of the above k-factorizations some factor is periodic. If any A_i is periodic the desired result holds. Thus it may be assumed that B and C are periodic. Let $B = H + B_1$, $C = K + C_1$, where H, K are non-zero subgroups of G and B_1, C_1 are not periodic. As G is cyclic, H and K cannot both be of order 2. Suppose that $k \in K$ and that $2k \neq 0$. Let H_1 denote the non-zero elements of H. Let $D = \{H_1, k\} + B_1$. Then

$$D + A_3 + A_4 + \ldots + A_{k+1} = \{H_1, k\} + B_1 + K + C_1 + \ldots + A_{k+1} =$$

$$H + B_1 + K + C_1 + \ldots + A_{k+1} = A_1 + A_2 + A_3 + A_4 + \ldots + A_{k+1} = G.$$

Now neither $\{H_1, k\}$ nor B_1 is periodic. As $\{H_1, k\} \subset H + K$ and $(H + K) + B_1$ is a direct sum, it follows from Lemma 2, that D is not periodic. But

$$G = D + A_3 + A_4 + \ldots + A_{k+1}$$

is a k-factorization of G. Hence some A_i is periodic. Therefore G has the Hajós $(k+1)$-property.

LEMMA 4. *If a group G possesses the Hajós k-property and a proper subgroup H of G admits $(k-1)$-factorizations then H possesses the Hajós $(k-1)$-property.*

PROOF. This follows immediately from Lemma 1.

2. Groups with the Hajós k-property

THEOREM 1. *In every factorization of a group G in which each factor has prime power order at least one factor is periodic.*

PROOF. Let G be represented as the additive group of integers modulo N. Let

$$G = A_1 + A_2 + \ldots + A_k$$

where each A_i has a prime power number of elements. Then

$$A_1(x) \cdot A_2(x) \ldots A_k(x) \equiv 1 + x + \ldots + x^{N-1} \left(\mathrm{mod}\ (x^N - 1)\right).$$

We may suppose without loss of generality that $F_N(x) \mid A_1(x)$. Let A_1 have p^m elements. Let $N = LM$ where $L = p^n$ and $p \nmid M$. Each polynomial $F_{p^r}(x)$ divides some $A_i(x)$, where $1 \leq r \leq n$. Since $F_{p^r}(1) = p$ and the polynomials concerned have integer coefficients and leading coefficients equal to one, it follows that p divides $A_i(1)$, which is the order of A_i. Further, as the different cyclotomic polynomials are relatively prime, it follows that if s different polynomials $F_{p^r}(x)$ divides $A_i(x)$ that p^s divides the order of A_i. It follows that if $A_i(1) = p^t$, not more than t such polynomials can divide $A_i(x)$. But as each of the n such polynomials divides some $A_i(x)$, and the product of the orders of the A_i is N, exactly t such polynomials must divide $A_i(x)$. As in the proof of Theorem 2 in Sands [8, p. 70—71] it may be deduced that no two integers occurring in A_i are congruent modulo L. In particular this holds for A_1.

Let $\varrho = \sigma\tau$ where σ and τ are L-th and M-th primitive roots of unity, respectively. Then ϱ is a primitive N-th root of unity and as $F_N(x) \mid A_1(x)$ it follows that $A_1(\varrho) = A_1(\sigma\tau) = 0$. Let $B(x) = A_1(x\tau)$. Then σ is a zero of $B(x)$, and the coefficients of $B(x)$ are from the field of M-th roots of unity. As L and M are relatively prime, $F_L(x)$ is irreducible over the field of M-th roots of unity. Therefore $F_L(x) \mid B(x)$. $F_L(x)$ also divides $(x^L - 1)$. Thus reducing $B(x)$ modulo $x^L - 1$, i.e. reducing the exponents of x modulo L, a polynomial $B_1(x)$ is obtained which is divisible by $F_L(x)$. The exponents of $B_1(x)$ are congruent to those of $B(x)$, and so to those of $A_1(x)$, modulo L. As the integers in A_1 are distinct modulo L it follows that the coefficients of $B_1(x)$ are zero or single powers of τ. Now $B_1(x)$ has degree less than L and $F_L(x) = 1 + x^{L/p} + \ldots + x^{(p-1)L/p}$. Thus the remaining factor has degree less than L/p. From the form of $B_1(x)$ and $F_L(x)$ the remaining factor has coefficients which are zero or single powers of τ. If $x^u\tau^v$ occurs in this factor then clearly $x^u\tau^v, x^{u+L/p}\tau^v, \ldots, x^{u+(p-1)L/p}\tau^v$ occur in $B_1(x)$. The corresponding set of p integers in A_1 are congruent to v modulo M and to $u, u + L/p, \ldots, u + (p-1)L/p$ modulo L. It follows that ML/p is a period of this set modulo N. Hence A_1 is periodic.

From this theorem it follows that cyclic groups of order p^n and groups of order N where $e(N) = k$ have the Hajós k-property. Note that the proof of the theorem shows that, if in any factorization of a group of order N, $F_N(x)$ divides a polynomial arising from a factor with a prime power number of elements, then this factor is periodic.

THEOREM 2. *If in a k-factorization of a group G every factor, except perhaps one, has order a power of a fixed prime p, then at least one factor is periodic.*

PROOF. This is a generalization of Theorem 2 of Sands [8], where the results was proved for the case $k = 2$. Using the results proved there the more general case follows readily. For if

$$G = A_1 + \ldots + A_{k-1} + A_k$$

where A_i has order a power of p, $1 \leqq i \leqq k-1$, then writing $A_1 + \ldots +$

$+ A_{k-1} = B$, B also has order a power of p and $G = B + A_k$. From the proof of Theorem 2 of Sands [8] it follows that if $F_N(x) \nmid B(x)$ then A_k is periodic. But if $F_N(x) \mid B(x)$ then, as $A_1(x) \ldots A_{k-1}(x) \equiv B(x) \pmod{x^N - 1}$, it follows that $F_N(x) \mid A_i(x)$ for some i, $1 \leq i \leq k - 1$. But A_i has a prime power number of elements and, as in Theorem 1, A_i must be periodic.

From this theorem it follows that, if G is a group of order $p^n q$, where p and q are distinct primes, G has the Hajós k-property.

3. All factorizations of groups with the Hajós k-property

The formulae given here are generalizations of the formulae given in Hajós [4] and Sands [8] for the case $k = 2$. If $G = A_1 + \ldots + A_k$, where the orders of the factors are as given in Theorem 1 or 2, then some factor, say A_1, is periodic. Thus $A_1 = H + B_1$ where H is a non-zero subgroup of G. Hence $G = H + B_1 + A_2 + \ldots + A_k$. This leads to a factorization of G/H as

$$G/H = (B_1 + H)/H + (A_2 + H)/H + \ldots + (A_k + H)/H.$$

Now again the orders of the factors will be as in Theorem 1 or 2 and so the process may be continued with some other factor being periodic. Using the notation $B \circ C$ to indicate any of the sets formed by adding to each element of B some element of C, the following formulae are eventually obtained:

$$A_1 = (\ldots((0 \overset{+}{\circ} H_1) \overset{+}{\circ} H_2) \ldots \overset{+}{\circ} H_{n-1}) \overset{+}{\circ} H_n$$

$$\ldots \qquad \ldots \qquad \ldots$$

$$\ldots \qquad \ldots \qquad \ldots$$

$$A_k = (\ldots((0 \overset{+}{\circ} H_1) \overset{+}{\circ} H_2) \ldots \overset{+}{\circ} H_{n-1}) \overset{+}{\circ} H_n$$

where at each position $\overset{+}{\circ} H_r$, one $+$ and $(k - 1)$ $\circ$ are used, and where $H_k + \ldots + H_n = K_k$ is a subgroup of G and $K_1 = G$. Here H_n is the subgroup H used above and the result is easily proved by induction on the order of G.

Every such chain of subgroups $G = K_1 \supset K_2 \supset \ldots \supset K_n$ gives rise to factorizations of G, and all such chains of subgroups and all sets of coset representatives H_r for K_r modulo K_{r-1} are known for finite cyclic groups. Thus these formulae effectively give all the factorizations arising from Theorems 1 and 2. In particular they give all factorizations of the groups which have been shown to have the Hajós k-property. In fact Lemmas 3 and 4 show that these properties are inherited by subgroups, and so by factor groups in this case, and thus the vital inductive step from G to G/H can always be taken. So all factorizations of all groups with the Hajós k-property arise in this way.

4. Groups not having the Hajós k-property

The results of this section apply to the case $k > 2$ only. It is shown that if $e(N) \geq k + 2$ and if N is not of the form p^n or $p^n q$ then there is a k-factorization of G in which no factor is periodic. By Lemmas 1 and 4 it suffices to prove this for the case $k = 3$. By Lemma 2 it suffices to prove the result for the border line case $e(N) = k + 2$. But for the case $k = 3$, $e(N) = 5$ the result has already been obtained by de Bruijn [1, Theorem 1]. The groups to be considered are those of order $p^3 q^2$, $p^3 qr$, $p^2 q^2 r$, $p^2 qrs$ and $pqrst$ where p, q, r, s and t are distinct primes. Each of these possesses a proper subgroup H which may be expressed as the direct product of subgroups of composite order. Thus de Bruijn's construction of non-periodic sets A and B such that[2] $AB = G$ goes through. But his set A is the direct product[2] of two non-periodic sets. Thus, changing back to the additive notation, G may be expressed as the direct sum of three non-periodic sets.

Note that taking G of order $p^3 q^2$ and H of order $p^2 q^2$, the three sets arising have orders p, q and $p^2 q$. Now extending this, using Lemma 1, by one factor of order p each time, to a k-factorization of the group of order $p^k q^2$, we obtain a factorization in which no factor is periodic, one factor having order $p^2 q$, one order q and the remaining $(k - 2)$ factors order p. From this we see that the condition in Theorem 1, that every factor shall have prime power order, cannot be weakened by omitting this condition for even one factor. We also see that in Theorem 2 the condition that at most one factor can have an order not a power of p cannot be weakened to allow two such factors, even where one of the exceptional factors has prime order. Thus, in this sense, Theorems 1 and 2 are as strong as can possibly be obtained.

5. The undecided case $e(N) = k + 1$

There remains the case where the exponent sum of the order of the group G exceeds the number of factors by one. If $k = 3$ this case follows from the solved case $k = 2$. For $k = 2$, groups of order N have the Hajós property for $e(N) = k + 2 = 4$ [8]. If $e(N) = 4$ and $G = A + B + C$, then two of the factors, say A and B, must have prime order. Let $A + B = D$. Then $G = C + D$. By the results obtained in [8] either C or D is periodic. If C is periodic, the desired result holds. If D is periodic then $D = H + D_1$, where H is a non-zero subgroup. Hence

$$A(x)\, B(x) \equiv D(x) \equiv H(x)\, D_1(x) \ \left(\mathrm{mod}\, (x^N - 1)\right).$$

[2] The multiplicative notation is used in de Bruijn's papers.

But for any non-zero subgroup H of G, $F_N(x) \mid H(x)$. Therefore $F_N(x)$ divides $A(x)$ or $B(x)$. But A and B have a prime number of elements. As in Theorem 1 it follows that A or B is periodic.

The Hajós k-property holds for groups of order $N = p^n q^m$, where $n + m = k + 1$ and p and q are distinct primes, for general k. This is proved using Theorem 1 of this paper or by a result of de Bruijn [2]. For if every factor has a prime power number of elements the result follows by Theorem 1. If not, then one factor, say A, has pq elements and the rest a prime number of elements. If $F_N(x) \nmid A(x)$ then again the result follows as in Theorem 1. But if $F_N(x) \mid A(x)$ then, by Theorem 2 of de Bruijn [2], we have

$$A(x) = \frac{x^N - 1}{x^{N/p} - 1} P(x) + \frac{x^N + 1}{x^{N/q} - 1} Q(x)$$

where $P(x)$ and $Q(x)$ have non-negative integral coefficients. But, substituting 1 for x, we have

$$pq = pP(1) + qQ(1), \text{ where } P(1) \geqq 0,\ Q(1) \geqq 0.$$

Hence either $P(1) = q$ and $Q(1) = 0$ or $P(1) = 0$ and $Q(1) = p$. But $P(1) = 0$ implies $P(x) = 0$ and $Q(1) = 0$ implies $Q(x) = 0$. Hence either $(x^N - 1)/(x^{N/p} - 1)$ or $(x^N - 1)/(x^{N/q} - 1)$ divides $A(x)$. Thus either N/p or N/q is a period of A.

But for general N and k the case $e(N) = k + 1$ remains undecided.

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QUOTIENT CATEGORIES AND QUASI-ISOMORPHISMS[1] OF ABELIAN GROUPS

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1. Introduction

The concept of quasi-isomorphism of Abelian groups was originated by B. Jónsson [13] who showed that if the notion of isomorphism is replaced by quasi-isomorphism, then one has a Krull-Schmidt theorem for torsion-free groups of finite rank. Indeed, the notion of quasi-isomorphism has come to play a significant role in the theory of Abelian groups. R. A. Beaumont and R. S. Pierce [4, 5, 6, 16] have utilized the concept rather extensively, in their work on torsion-free rings, and J. D. Reid [17, 18] has investigated the relation between the structure of a torsion-free Abelian group and the structure of its ring of quasi-endomorphisms. The basic intention in this paper is to provide a natural setting for the study of quasi-isomorphisms. It is shown that the notion of quasi-isomorphism of torsion-free groups is a natural one; specifically, that two torsion-free Abelian groups are quasi-isomorphic if and only if they are isomorphic in the quotient category $\mathscr{A}/\mathscr{B}$, where $\mathscr{A}$ is the category of all Abelian groups and $\mathscr{B}$ is the class of all bounded Abelian groups. In the setting $\mathscr{A}/\mathscr{B}$, Jónsson's Krull-Schmidt theorem for quasi-decompositions, as well as Reid's generalizations of it become category theory theorems, and lend themselves to generalizations from groups to modules over suitable rings. Furthermore, a question of L. Fuchs concerning the quasi-splitting of an Abelian group over its torsion subgroup is easily answered, once translated to the category $\mathscr{A}/\mathscr{B}$.

The category $\mathscr{A}/\mathscr{B}$ has some deficiencies. For example, it does not have injective envelopes and does not have infinite direct sums. This situation can be remedied by embedding small Abelian subcategories $\mathscr{C}$ of $\mathscr{A}/\mathscr{B}$ in the category $\mathscr{L}(\mathscr{C}, \mathscr{A})$ of all left exact functors from $\mathscr{C}$ to $\mathscr{A}$. The final section is a collection of miscellaneous, but perhaps interesting, facts concerning such an embedding. For example, in the larger category

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$\mathscr{L}(\mathcal{C}, \mathscr{A})$, a group has no new direct decompositions, but may gain some new subobjects. The significance of these new subobjects is not too clear. The fact that, in $\mathscr{L}(\mathcal{C}, \mathscr{A})$, extensions of representable functors are representable may be worthy of note. Finally, it is remarked that $\mathscr{L}(\mathcal{C}, \mathscr{A})$ is the same as the category of all left exact functors from $\mathcal{C}$ to the category of modules over the ring Q of rational numbers.

An occasional detail of a proof is suppressed since a more extensive paper on quotient categories of modules is in preparation. However, ample indications of all proofs are provided and the reader will have no difficulty filling in the occasional missing steps.

2. Preliminaries

The term *Abelian category* will be used in the sense of MacLane [14], and $\mathscr{A}$ will always denote the category of all Abelian groups. The word *group* will mean Abelian group. For A, B in an Abelian category $\mathcal{C}$, $\mathrm{Hom}_{\mathcal{C}}(A, B)$ will denote the group of maps (or morphisms) from A to B, except that $\mathrm{Hom}_{\mathscr{A}}(A, B)$ will be written simply as $\mathrm{Hom}(A, B)$. A non-empty subclass $\mathscr{S}$ of $\mathscr{A}$ is a *Serre class* of $\mathscr{A}$ if for every exact sequence

$$0 \to A \to B \to C \to 0$$

of groups, B is in $\mathscr{S}$ if and only if A and C are in $\mathscr{S}$. Equivalently, $\mathscr{S}$ is a non-empty class of groups closed under subgroups, homomorphic images, and extensions. Note in particular that if $A \in \mathscr{S}$, then every group isomorphic to A is in $\mathscr{S}$. From a Serre class $\mathscr{S}$ of $\mathscr{A}$, there arises the *quotient category* $\mathscr{A}/\mathscr{S}$, as defined by Grothendieck [9]. The objects of $\mathscr{A}/\mathscr{S}$ are just the objects of $\mathscr{A}$. To define the maps of $\mathscr{A}/\mathscr{S}$, let $A, B \in \mathscr{A}$, and A', B' be subgroups of A and B, respectively. There is a natural homomorphism

$$\mathrm{Hom}(A, B) \to \mathrm{Hom}(A', B/B').$$

Considering all subgroups A' and B' such that A/A' and B' are in $\mathscr{S}$, one has a direct system, and $\mathrm{Hom}_{\mathscr{A}/\mathscr{S}}(A, B)$ is defined to be the group $\varinjlim_{A', B'} \mathrm{Hom}(A', B/B')$. To define composition of maps in $\mathscr{A}/\mathscr{S}$, let $\bar{f} \in \mathrm{Hom}_{\mathscr{A}/\mathscr{S}}(A, B)$, $\bar{g} \in \mathrm{Hom}_{\mathscr{A}/\mathscr{S}}(B, C)$. Then $\bar{f}$ comes from an $f \in \mathrm{Hom}(A', B/B')$, and $\bar{g}$ comes from a $g \in \mathrm{Hom}(B'', C/C')$, with A/A', B', B/B'', $C' \in \mathscr{S}$. If $A'' = f^{-1}((B'' + B')/B')$, then $A/A'' \in \mathscr{S}$, and f induces a map $f' \in \mathrm{Hom}(A'', (B'' + B')/B')$. If $C''/C' = g(B'' \cap B')$, then $C'' \in \mathscr{S}$, and g induces a map $g' \in \mathrm{Hom}(B''/(B'' \cap B'), C/C'')$. Now let h be the composition of the maps

$$A'' \xrightarrow{f'} (B'' + B')/B' \approx B''/(B'' \cap B') \xrightarrow{g'} C/C''.$$

Since A/A'' and C'' are in $\mathscr{S}$, h determines an element $\bar{h} \in \mathrm{Hom}_{\mathscr{A}/\mathscr{S}}(A, C)$. It is straightforward that $\bar{h}$ is uniquely determined by $\bar{f}$ and $\bar{g}$ and that $\bar{g} \circ \bar{f} = \bar{h}$ defines a bilinear composition. With these definitions, $\mathscr{A}/\mathscr{S}$ becomes an Abelian category. (For a proof of this latter fact, see [8].) The following elementary facts are needed in the sequel. They follow easily from the definition of $\mathscr{A}/\mathscr{S}$.

Let $\bar{f}$ denote the element of $\mathrm{Hom}_{\mathscr{A}/\mathscr{S}}(A,B)$ determined by $f \in \mathrm{Hom}(A,B)$. Then

2.1. The functor J from $\mathscr{A}$ to $\mathscr{A}/\mathscr{S}$ defined by $J(A) = A$ and $J(f) = \bar{f}$ is exact with kernel $\mathscr{S}$. This canonical functor from $\mathscr{A}$ to $\mathscr{A}/\mathscr{S}$ will always be denoted by J.

Let $f' \in \mathrm{Hom}(A', B/B')$, $g' \in \mathrm{Hom}(A'', B/B'')$ with A/A', A/A'', B', $B'' \in \mathscr{S}$. Then f' and g' determine maps $\bar{f}', \bar{g}' \in \mathrm{Hom}_{\mathscr{A}/\mathscr{S}}(A, B)$, and

2.2. $\bar{f}' = 0$ if and only if $\mathrm{Im}\, f' \in \mathscr{S}$.

2.3. $\bar{f}'$ is an epimorphism if and only if $\mathrm{Coker}\, f' \in \mathscr{S}$.

2.4. $\bar{f}'$ is a monomorphism if and only if $\mathrm{Ker}\, f' \in \mathscr{S}$.

2.5. $\bar{f}' = \bar{g}'$ if and only if there exist subgroups S of A and T of B such that

(a) $S \subset A' \cap A''$

(b) $A/S \in \mathscr{S}$

(c) $B'' + B' \subset T$

(d) $T \in \mathscr{S}$

(e) The maps from S to B/T induced by f' and g' are the same.

For any Abelian category $\mathscr{C}$, the functors $\mathrm{Ext}^n_{\mathscr{C}}$ are defined [14], and the *dimension* of $\mathscr{C}$, denoted $\dim(\mathscr{C})$ is the smallest integer n, if such exists, such that $\mathrm{Ext}^{n+1}_{\mathscr{C}}(A, B) = 0$ for all $A, B \in \mathscr{C}$. If no such n exists, $\dim(\mathscr{C}) = \infty$. It is significant that for the category $\mathscr{A}$ and any Serre class $\mathscr{S}$, $\dim(\mathscr{A}/\mathscr{S}) \leq \leq 1$. In fact,

PROPOSITION 2.6. *If*

$$0 \to K \xrightarrow{f} P \xrightarrow{g} A \to 0$$

is a projective resolution of A in the category $\mathscr{A}$, then

$$0 \to J(K) \xrightarrow{J(f)} J(P) \xrightarrow{J(g)} J(A) \to 0$$

is a projective resolution of $J(A)$ in the category $\mathscr{A}/\mathscr{S}$. In particular, if A is projective in $\mathscr{A}$, then $J(A)$ is projective in $\mathscr{A}/\mathscr{S}$.

PROOF. Consider the diagram

$$\begin{array}{ccc} & & P \\ & & \big\downarrow{\scriptstyle \bar{g}} \\ A & \xrightarrow{\bar{f}} & B \to 0 \end{array}$$

in $\mathscr{A}/\mathscr{S}$, with exact row. Now $\bar{f}$ is associated with a (not necessarily unique) homomorphism $f' \in \mathrm{Hom}(A', B/B')$, $\bar{g}$ with $g' \in \mathrm{Hom}(P', B/B'')$, where $A/A', P/P', B', B'' \in \mathcal{S}$. Let $S = B' + B''$. Then $S \in \mathcal{S}$, and, in $\mathcal{A}$, there results the diagram

$$\begin{array}{ccc} & & P' \\ & & \downarrow{\scriptstyle g} \\ A' & \xrightarrow{f} & B/S \end{array}$$

where f is the composition of f' with the natural map from B/B' to B/S, and g is defined similarly. Let $C/S = \mathrm{Im}\, f$, and $P'' = g^{-1}(C/S)$. Then $(B/S)/(C/S)$ and $P/P'' \in \mathcal{S}$, and the diagram

$$\begin{array}{ccccc} & & P'' & & \\ & & \downarrow{\scriptstyle g_1} & & \\ A' & \xrightarrow{f_1} & C/S & \longrightarrow & 0 \end{array}$$

in $\mathcal{A}$, with the obvious maps f_1 and g_1, has exact row. But P'' is projective in $\mathcal{A}$ (since the projectives in $\mathcal{A}$ are just the free groups), and so there exists a map $h : P'' \to A'$ such that

$$\begin{tikzcd} & P'' \arrow[dl, "h"'] \arrow[d, "g_1"] \\ A' \arrow[r, "f_1"] & B/S \end{tikzcd}$$

is commutative. Going over to $\mathcal{A}/\mathcal{S}$, f_1 and g_1 determine maps $\bar{f}_1 \in \in \mathrm{Hom}_{\mathcal{A}/\mathcal{S}}(A, B)$ and $\bar{g}_1 \in \mathrm{Hom}_{\mathcal{A}/\mathcal{S}}(P, B)$. Further, $\bar{f}_1 = \bar{f}, \bar{g}_1 = \bar{g}$, and the diagram

$$\begin{tikzcd} & P \arrow[dl, "\bar{h}"'] \arrow[d, "\bar{g}"] & \\ A \arrow[r, "\bar{f}"] & B \arrow[r] & 0 \end{tikzcd}$$

is commutative. That is, P is projective in $\mathcal{A}/\mathcal{S}$. Using the exactness of J, the proposition follows readily.

In a dual fashion, one can show that J takes injective resolutions in $\mathcal{A}$ into injective resolutions in $\mathcal{A}/\mathcal{S}$. That $\mathrm{Ext}^n_{\mathcal{A}/\mathcal{S}} = 0$ for $n \geq 2$ follows from Proposition 2.6. Hence $\dim(\mathcal{A}/\mathcal{S}) \leq 1$.

Proposition 2.7. *For any Serre class $\mathcal{S}$ of $\mathcal{A}$, P is projective (injective) in $\mathcal{A}/\mathcal{S}$ if and only if P is isomorphic in $\mathcal{A}/\mathcal{S}$ to P_1 where P_1 is projective (injective) in $\mathcal{A}$.*

Proof. Let P be projective in $\mathcal{A}/\mathcal{S}$ and $F \xrightarrow{f} P \to 0$ exact in $\mathcal{A}$ with F free. In $\mathcal{A}/\mathcal{S}$, the sequence $F \xrightarrow{\bar{f}} P \to 0$ splits, so that there is a map $\bar{g} : P \to F$ in $\mathcal{A}/\mathcal{S}$ with $\bar{g}$ a monomorphism. This yields a map $g : P' \to F/F'$ in $\mathcal{A}$, with P/P' and $F' \in \mathcal{S}$. Since $\bar{g}$ is a monomorphism in $\mathcal{A}/\mathcal{S}$, $\mathrm{Ker}\, g \in \mathcal{S}$, and in $\mathcal{A}$, $P'/\mathrm{Ker}\, g \cong \mathrm{Im}\, g = F''/F'$ for some subgroup F'' of F. It follows that in $\mathcal{A}/\mathcal{S}$, $P \cong P' \cong P'/\mathrm{Ker}\, g \cong \mathrm{Im}\, g \cong F''$ which is projective in $\mathcal{A}$ as it is a free group. The rest of the proof is clear.

Following Hu [11], a Serre class $\mathscr{S}$ is *strongly complete* if $\mathscr{S}$ is closed under arbitrary infinite direct sums, and is *complete* if $A \in \mathscr{S}$, $B \in \mathscr{A}$ implies $A \otimes B \in \mathscr{S}$. Equivalently, $\mathscr{S}$ is complete if for each $A \in \mathscr{S}$, $(\sum_{a \in I} A_a) \in \mathscr{S}$ whenever $A_a \cong A$ for each $a \in I$. A complete Serre class $\mathscr{S}$ is *bounded* if every group in $\mathscr{S}$ is bounded. The strongly complete and the bounded complete Serre classes are very easily determined.[2] Let M be a multiplicatively closed subset of the ring Z of integers. Recall that M is *saturated* if $mn \in M$ implies that $m \in M$. (The possibility that $0 \in M$ is not excluded.) Denote by Z_M the ring of quotients determined by M. The following two propositions are of particular interest because of the generalizations they admit [8, 20].

PROPOSITION 2.8. *Let $\mathscr{I}(M)$ be the class of those groups A such that $Z_M \otimes A = 0$. Then $M \to \mathscr{I}(M)$ is a natural one-one correspondence between the saturated multiplicatively closed subsets of Z and the strongly complete Serre classes of $\mathscr{A}$.*

PROOF. That $\mathscr{I}(M)$ is a strongly complete Serre class is trivial. In fact, $\mathscr{I}(M)$ consists of the groups each element of which is annihilated by some element of M. A saturated multiplicatively closed set is obtained from a strongly complete Serre class $\mathscr{S}$ by taking the generators of the annihilators of the cyclic groups in $\mathscr{S}$, and this is the inverse of the correspondence $M \to \mathscr{I}(M)$.

It is not difficult to show that Z_M is the endomorphism ring of Z in the category $\mathscr{A}/\mathscr{I}(M)$.

PROPOSITION 2.9. *Let $\mathscr{B}(M)$ be the class of those groups A such that $Z_M \otimes \operatorname{Hom}(A, A) = 0$. Then $M \to \mathscr{B}(M)$ is a natural one-one correspondence between the saturated multiplicatively closed subsets of Z and the bounded complete Serre classes of $\mathscr{A}$.*

PROOF. Again, $\mathscr{B}(M)$ is easily seen to be a bounded complete Serre class, and M is recovered from a bounded complete Serre class $\mathscr{S}$ by taking the generators of the annihilators of the cyclic groups in $\mathscr{S}$.

In this case also, Z_M is the endomorphism ring of Z in the quotient category $\mathscr{A}/\mathscr{B}(M)$.

The quotient categories $\mathscr{A}/\mathscr{I}(M)$ are nothing new. In fact, $\mathscr{A}/\mathscr{I}(M)$ is equivalent to the category of all Z_M-modules, the equivalence being given by the functor $F(A) = Z_M \otimes A$ from $\mathscr{A}/\mathscr{I}(M)$ to the category of Z_M-modules. On the other hand, the categories $\mathscr{A}/\mathscr{B}(M)$ are not equivalent to categories of all modules over a ring.

[2] All Serre classes of Abelian groups have been determined by S. Balcerzyk. Although E. James Peake, Jr. [15] has pointed out that the crucial Lemma 1 in [3] is incorrect, Balcerzyk has repaired the damage.

3. The category $\mathscr{A}/\mathscr{B}$

From now on, attention will be restricted to the category $\mathscr{A}/\mathscr{B}$, where $\mathscr{B}$ is the Serre class of all bounded groups. However, much of what is said is true for $\mathscr{A}/\mathscr{S}$, when $\mathscr{S}$ is any bounded complete Serre class.

Two Abelian categories $\mathscr{C}$ and $\mathscr{D}$ are *equivalent* if there exists a (covariant) functor F from $\mathscr{C}$ to $\mathscr{D}$ such that for $A, B \in \mathscr{C}$, the map $\mathrm{Hom}_{\mathscr{C}}(A, B) \to \mathrm{Hom}_{\mathscr{D}}(F(A), F(B))$ induced by F is an isomorphism, and each $D \in \mathscr{D}$ is isomorphic in $\mathscr{D}$ to an object of the form $F(C)$, $C \in \mathscr{C}$. The functor F is called an *equivalence*. This relation is an equivalence relation, and equivalent categories have the same homological properties [8].

Now from the ring Q and the category $\mathscr{A}$, a new category $\mathscr{A}_Q$ is defined as follows. The objects of $\mathscr{A}_Q$ are just the Abelian groups. The group $\mathrm{Hom}_{\mathscr{A}_Q}(A, B)$ is the group $Q \otimes_Z \mathrm{Hom}(A, B)$ (written simply $Q \otimes \mathrm{Hom}(A, B)$). Composition of maps in $\mathscr{A}_Q$ is given by $(r \otimes f) \circ (s \otimes g) = (rs) \otimes (f \circ g)$. Every element of $Q \otimes \mathrm{Hom}(A, B)$ can be written in the form $1/n \otimes f$ with $n > 0$, and a functor F from $\mathscr{A}_Q$ to $\mathscr{A}/\mathscr{B}$ is defined as follows. Let $F(A) = A$. For a positive integer n and group A, let α_n be the natural isomorphism $\alpha_n : nA \to A/A[n]$. For $f \in \mathrm{Hom}(A, B)$, let f_n be the map $A/A[n] \to B/B[n]$ induced by f. Now $f_n \circ \alpha_n$ maps nA to $B/B[n]$ and hence induces a map $\overline{f_n \circ \alpha_n} \in \mathrm{Hom}_{\mathscr{A}/\mathscr{B}}(A, B)$. Define $F(1/n \otimes f) = \overline{f_n \circ \alpha_n}$.

THEOREM 3.1. *The functor F is an equivalence between the categories $\mathscr{A}_Q$ and $\mathscr{A}/\mathscr{B}$.*

PROOF. It is straightforward to check that F is an additive functor. For the remainder of the proof, it suffices to show that $F : Q \otimes \mathrm{Hom}(A, B) \to \mathrm{Hom}_{\mathscr{A}/\mathscr{B}}(A, B)$ is bijective. Let $\bar{g} \in \mathrm{Hom}_{\mathscr{A}/\mathscr{B}}(A, B)$. Then $\bar{g}$ comes from a map $g : A' \to B/B'$ with $A/A', B' \in \mathscr{B}$. By 2.5 it may be assumed that $A' = mA$, $B' = B[m]$ for some positive integer m. Let β_m be the usual isomorphism $B/B[m] \to mB$. Now define $G(\bar{g}) = 1/m^2 \otimes (\beta_m \circ g \circ m) \in Q \otimes \mathrm{Hom}(A, B)$. Then $F(G(\bar{g})) = F(1/m^2 \otimes (\beta_m \circ g \circ m)) = \overline{(\beta_m \circ g \circ m)_{m^2} \circ \alpha_{m^2}}$. Let $p : A \to A/A[m^2]$, $\pi : B/B[m] \to B/B[m^2]$, and $q_n : B \to B/B[n]$ be the natural epimorphisms. Then $(\beta_m \circ g \circ m)_{m^2}$ is the unique map such that

$$\begin{array}{ccc} A & \xrightarrow{p} & A/A[m^2] \\ {\scriptstyle q_{m^2} \circ \beta_m \circ m}\downarrow & \swarrow {\scriptstyle (\beta_m \circ g \circ m)_{m^2}} & \\ B/B[m^2] & & \end{array}$$

commutes. Let g' be the restriction of g to $m^2 A$ followed by π. Then $g' \circ \alpha_{m^2}^{-1} \circ p = g' \circ m^2 = \pi \circ g \circ m^2 = \pi \circ m \circ g \circ m = \pi \circ (q_m \circ \beta_m) \circ g \circ m = q_{m^2} \circ \beta_m \circ g \circ m$. Thus $(\beta_m \circ g \circ m)_{m^2} = g' \circ \alpha_{m^2}^{-1}$, and $F(G(\bar{g})) = \overline{(\beta_m \circ g \circ m)_{m^2} \circ \alpha_{m^2}} = \overline{(g' \circ \alpha_{m^2}^{-1}) \circ \alpha_{m^2}} = \bar{g}'$. But g and g' determine the same element of $\operatorname{Hom}_{\mathcal{A}/\mathcal{B}}(A, B)$; that is, $\bar{g} = \bar{g}'$, or $F(G(\bar{g})) = \bar{g}$. On the other hand, $G(F(1/m \otimes g)) = G(\overline{g_m \circ \alpha_m}) = 1/m^2 \otimes (\beta_m \circ g_m \circ \alpha_m \circ m) = 1/m^2 \otimes mg = 1/m \otimes g$, and this completes the proof.

From now on, the categories $\mathcal{A}_Q$ and $\mathcal{A}/\mathcal{B}$ will be identified, and the notation $\mathcal{A}_Q$ dropped. That is, $\operatorname{Hom}_{\mathcal{A}/\mathcal{B}}(A, B)$ is identified with $Q \otimes \operatorname{Hom}(A, B)$.

Let n be a positive integer and $f \in \operatorname{Hom}(A, B)$. The following, and other similar statements are readily verified.

3.2. (a) $1/n \otimes f$ is an epimorphism if and only if $\operatorname{Coker} f \in \mathcal{B}$.

(b) $1/n \otimes f$ is a monomorphism if and only if $\operatorname{Ker} f \in \mathcal{B}$.

(c) A kernel of $1/n \otimes f$ is $1 \otimes (\ker f)$.

(d) An image of $1/n \otimes f$ is $1 \otimes (\operatorname{im} f)$.

(e) A direct sum of A and B in $\mathcal{A}/\mathcal{B}$ is $J(A \oplus B)$.

(f) The sequence $A \xrightarrow{1/m \otimes f} B \xrightarrow{1/n \otimes g} C$ is exact if and only if

$$(\operatorname{Im} f + \operatorname{Ker} g)/((\operatorname{Im} f) \cap (\operatorname{Ker} g)) \in \mathcal{B}.$$

(g) $J(f) = 1 \otimes f$.

The value of Theorem 3.1 lies in the fact that it makes certain computations in $\mathcal{A}/\mathcal{B}$ rather easy. In particular, $\operatorname{Ext}_{\mathcal{A}/\mathcal{B}}(A, B)$ is readily determined.

Corollary 3.3. $\operatorname{Ext}_{\mathcal{A}/\mathcal{B}}(A, B) \cong Q \otimes \operatorname{Ext}(A, B)$.

Proof. An injective resolution

$$0 \to B \xrightarrow{f} I \xrightarrow{g} I/B \to 0$$

in $\mathcal{A}$ yields an injective resolution

$$0 \to B \xrightarrow{1 \otimes f} I \xrightarrow{1 \otimes g} I/B \to 0$$

in $\mathcal{A}/\mathcal{B}$. There result the exact sequences

$$0 \to \operatorname{Hom}(A, B) \to \operatorname{Hom}(A, I) \to \operatorname{Hom}(A, I/B) \to \operatorname{Ext}(A, B) \to 0,$$

$$0 \to Q \otimes \operatorname{Hom}(A, B) \to Q \otimes \operatorname{Hom}(A, I) \to Q \otimes \operatorname{Hom}(A, I/B) \to$$
$$\to Q \otimes \operatorname{Ext}(A, B) \to 0,$$

and

$$0 \to Q \otimes \operatorname{Hom}(A, B) \to Q \otimes \operatorname{Hom}(A, I) \to Q \otimes \operatorname{Hom}(A, I/B) \to$$
$$\to \operatorname{Ext}_{\mathcal{A}/\mathcal{B}}(A, B) \to 0.$$

It follows that $Q \otimes \operatorname{Ext}(A, B) \cong \operatorname{Ext}_{\mathcal{A}/\mathcal{B}}(A, B)$ as asserted. The isomorphism is clearly functorial.

The following is immediate.

Corollary 3.4. *If $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is exact in $\mathcal{A}$, then $0 \to A \xrightarrow{1 \otimes f} B \xrightarrow{1 \otimes g} C \to 0$ splits (in $\mathcal{A}/\mathcal{B}$) if and only if $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ represents an element of finite order in* $\operatorname{Ext}(C, A)$.

Corollary 3.5. *In $\mathscr{A}/\mathscr{B}$, P is projective if and only if, in $\mathscr{A}$, $P = F \oplus B$ with F free and B bounded, and I is injective if and only if, in $\mathscr{A}$, $I = D \oplus B$ with D divisible and B bounded.*

Proof. Let P be projective in $\mathscr{A}/\mathscr{B}$. Then $Q \otimes \mathrm{Ext}(P, X) = 0$ for all $X \in \mathscr{A}$, so that $\mathrm{Ext}(P, X)$ is torsion for all X. But $\mathrm{Ext}(P, X)$ is the direct sum of a cotorsion group [10] and a divisible group, so is the direct sum of a bounded and a divisible group. Denoting the torsion subgroup of P by P_t, one has $\mathrm{Ext}(P_t, X) \cong \mathrm{Ext}(\mathrm{Tor}(Q/Z, P), X) \cong \mathrm{Ext}\big(P, \mathrm{Ext}(Q/Z, X)\big)$ is reduced, and hence bounded. Thus P $\mathrm{Ext}(P_t, X) = 0$ for all X, so that $P_t = \sum_a C_a$, C_a cyclic. Now $\mathrm{Ext}(P_t, P_t) \cong \prod_a \mathrm{Ext}(C_a, P_t)$, which contains a copy of $\prod_a C_a$. Thus $\sum_a C_a = P_t$ is bounded and $P = P_t \oplus P_1$. Let $0 \to K \to F \to P_1 \to 0$ be exact with F free. This sequence represents in $\mathrm{Ext}(P_1, K)$ an element of finite order n, and by Theorem 1 in Walker [19], $0 \to K \to K + nF \to nP_1 \to 0$ splits. Hence nP_1 is isomorphic to a summand of the free group $K + nF$, so is itself free. It follows that P_1 is free and P has the desired form. Clearly $P = F \oplus B$, with F free and B bounded, is projective in $\mathscr{A}/\mathscr{B}$.

Let I be injective in $\mathscr{A}/\mathscr{B}$. Then $Q \otimes \mathrm{Ext}(X, I) = 0$ for all X so that $\mathrm{Ext}(X, I)$ is torsion for all X. In particular, $\mathrm{Ext}(Q, I)$ is torsion. But it is torsion-free and hence 0. Thus I is the direct sum of a cotorsion and a divisible group. ($Ext(Q/Z, I)$ is torsion and cotorsion, whence bounded and isomorphic to I/I_d, where I_d is the maximum divisible subgroup of I. It follows that I has the desired form. Clearly $I = D \oplus B$, with D divisible and B bounded, is injective in $\mathscr{A}/\mathscr{B}$.

4. Applications

Two groups A and B are *quasi-isomorphic* if there exist isomorphic subgroups S and T of A and B respectively, with A/S, $B/T \in \mathscr{B}$. For torsion-free groups, this is equivalent to each being isomorphic to a subgroup of the other with bounded quotients. From the definition of $\mathscr{A}/\mathscr{B}$, it is immediate that *two torsion-free groups are quasi-isomorphic if and only if they are isomorphic in $\mathscr{A}/\mathscr{B}$.* Furthermore, quasi-endomorphisms, quasi-decompositions, etc., as defined for example by Reid [18], are just endomorphisms, decompositions, etc., in the category $\mathscr{A}/\mathscr{B}$. In fact Reid himself points out [18] that his ring of quasi-endomorphisms of a torsion-free group G is $Q \otimes \mathrm{Hom}(G, G)$. Thus the decomposition theory of torsion-free groups in $\mathscr{A}/\mathscr{B}$ is equivalent to the quasi-decomposition theory of torsion-free groups in $\mathscr{A}$. The advantage of the setting $\mathscr{A}/\mathscr{B}$ besides its naturalness is that the homological algebra of $\mathscr{A}/\mathscr{B}$, and category theory in general, is available for application.

The definition of quasi-isomorphism given above is applied to all groups (not just torsion-free groups), and if this is done, quasi-isomorphism (say of two torsion groups,) is not equivalent to isomorphism in $\mathscr{A}/\mathscr{B}$. Even though quasi-isomorphism in this sense may turn out to be valuable, it does not seem to be natural. It is perhaps significant that quasi-isomorphism and quasi-decomposition theory has been of principal value only for torsion-free groups. Thus it is submitted that the proper definition of quasi-isomorphism should be isomorphism in $\mathscr{A}/\mathscr{B}$.

As applications, two theorems (neither new) are quickly proved. It is hoped that the methods (namely, the utilization of the setting $\mathscr{A}/\mathscr{B}$, or more generally, $\mathscr{A}/\mathscr{S}$ for appropriate Serre classes $\mathscr{S}$) will be of further value.

THEOREM 4.1. *In $\mathscr{A}/\mathscr{B}$, suppose, $A = \sum_{i=1}^{m} A_i$ with the endomorphism ring of each A_i local. Then any two decompositions of A into indecomposable summands are equivalent. Furthermore, any two decompositions of A have equivalent refinements.*

PROOF. This theorem holds in any Abelian category. The proof given here is modeled after the proof of Lemma 1 and Theorem 1 in Azumaya [1]. Let $A = \sum_{\alpha \in I} B_\alpha$, with B_α indecomposable. (Keep in mind that the setting is in $\mathscr{A}/\mathscr{B}$.) Let $\{f_\alpha\}_{\alpha \in I}$ be a set of mutually orthogonal idempotents corresponding to this decomposition, and let $\{e_1, \ldots, e_m\}$ be a similar set for the decomposition $\sum_{i=1}^{m} A_i$. For $\alpha \in I$, $1 = f_\alpha + (1 - f_\alpha)$, and $e_1 = e_1 f_\alpha + {}$ $+ e_1(1 - f_\alpha)$. On A_1, e_1 is the identity map, so either $e_1 f_\alpha$ or $e_1(1 - f_\alpha)$ is an automorphism of A_1, the endomorphism ring of A_1 being local. So, either f_α or $1 - f_\alpha$ maps A_1 isomorphically onto a subobject B_1 of A. Now e_1 maps B_1 isomorphically onto A_1, and on A, e_1 has kernel $\sum_{i=2}^{m} A_i$. It follows that $A = B_1 \oplus \sum_{i=2}^{m} A_i$. Now let $\{e_1', e_2', \ldots, e_m'\}$ be a set of mutually orthogonal idempotents for this last decomposition. As before, $e_2' f_\alpha$ or $e_2'(1 - f_\alpha)$ induces an automorphism of A_2, and either f_α or $1 - f_\alpha$ maps A_2 onto a subobject B_2 such that $A = B_1 \oplus B_2 \oplus \sum_{i=3}^{m} A_i$. Continuing in this manner, one gets $A = \sum_{i=1}^{m} B_i$, with B_i either $f_\alpha(A_i)$ or $(1 - f_\alpha)(A_i)$. But $1 - f_\alpha$ cannot map A_i isomorphically onto B_i for all i unless $B_\alpha = 0$. It may be assumed that each $f_\alpha \neq 0$. It follows from the hypothesis that, for some i, $A_i \cong B_\alpha$. But each B_α is indecomposable, and $f_\alpha(A_i) = B_i$ is a summand of A and is contained in $f_\alpha(A) = B_\alpha$. Thus $B_i = B_\alpha \cong A_i$. Now $A = B_\alpha \oplus \sum_{i \neq j} A_j$. Suppose $\beta \neq \alpha$. Applying what was just done to this decomposition and to the decomposition $\sum_{\alpha \in I} B_\alpha$, one gets that f_β is an isomorphism from some summand of the decomposition $A = B_a \oplus \sum_{i \neq j} A_j$ to B_β, and it cannot be from B_a. Thus there exists $k \neq i$ such that $A = B_a \oplus B_\beta \oplus \sum_{i,k \neq j} A_j$. Continuing, the B_a and the A_i must be exhausted at the same step. The first assertion of the theorem follows. To prove the second assertion, note that for any idempotent f, there is an i such that f maps A_i isomorphically onto a summand of A; in particular onto a summand of $f(A)$. Thus every summand of A has an indecomposable summand. A procedure similar to the proof of the first assertion yields the desired result.

Note that any object in an Abelian category which has local endomorphism ring is indecomposable. It is not difficult to show in the setting $\mathscr{A}/\mathscr{B}$ that a torsion-free indecomposable group of finite rank has local endomorphism ring. Thus Theorem 4.1. implies Jónsson's theorem. Further, Reid's Theorems 4.1, 4.2, 4.3, 4.4, and Corollary 4.3 [18] may be interpreted as results showing certain connections between properties of the endomorphism ring of an object and decompositions (finite) of that object, and hold in any Abelian category.

A group that splits in $\mathscr{A}$ over its torsion subgroup is called a *splitting* group. L. Fuchs has asked whether the quasi-isomorphism (in the sense of the definition at the beginning of Section 4) of a group G with a splitting group implies that G is a splitting group. This question is now easily handled with the aid of

LEMMA 4.2. *The group G is quasi-isomorphic to a splitting group if and only if G is isomorphic in $\mathscr{A}/\mathscr{B}$ to a splitting group.*

PROOF. Suppose $\bar{g} : G \to T \oplus A$ is an isomorphism in $\mathscr{A}/\mathscr{B}$, the direct sum taken in $\mathscr{A}$, T torsion, and A torsion-free. Then for some positive integers m and n, and $B \in \mathscr{B}$, g induces a map $g : nG \to T/B \oplus A$ in $\mathscr{A}$, with $\operatorname{Ker} g \subset (mG)[n]$ and $\operatorname{Coker} g \in \mathscr{B}$. Let $S/B = g\big((mG)[n]\big)$. Then the composition of the maps (in $\mathscr{A}$) $mnG \cong (mG)/(mG)[n] \xrightarrow{g'} T/S \oplus A$, g' induced by g, has bounded cokernel, so that G is quasi-isomorphic to a splitting group, g' being a monomorphism. Conversely, if in $\mathscr{A}$, $nG \xrightarrow{f} T \oplus A$ is a monomorphism with bounded cokernel, then f induces an isomorphism in $\mathscr{A}/\mathscr{B}$.

Call a group *quasi-splitting* if it is quasi-isomorphic to a splitting group. To find a quasi-splitting group that is not a splitting group, it then suffices to find an exact sequence $0 \to T \to G \to A \to 0$ in $\mathscr{A}$, with T torsion and A torsion-free, that represents a non-zero element of finite order in $\operatorname{Ext}(A, T)$. These statements follow from Lemma 4.2 and Corollary 3.3. The existence of such a sequence is established in Baer [2], so the answer to Fuchs' question is no. If, for example, A is countable, then $\operatorname{Ext}(A, T)$ is torsion-free, so that in this case the quasi-splitting of G is equivalent to the splitting of G. More detailed information concerning this question can be found in Walker [19].

Nothing non-trivial is known about decompositions of p-groups in $\mathscr{A}/\mathscr{B}$. For example, it is not known whether there are p-groups of arbitrarily large final rank that are indecomposable in $\mathscr{A}/\mathscr{B}$. This question is related to one posed by Pierce [12]. A general study of decompositions of groups in $\mathscr{A}/\mathscr{S}$, with $\mathscr{S}$ well-chosen Serre classes, would seem being worthwhile.

5. Embeddings of small subcategories of $\mathscr{A}/\mathscr{B}$

The category $\mathscr{A}/\mathscr{B}$ has several homological deficiencies. Although it has enough projectives and injectives, $B = \sum_{i=1}^{\infty} C(p^i)$, for example, does not have an injective envelope in $\mathscr{A}/\mathscr{B}$. Further, infinite sums in $\mathscr{A}/\mathscr{B}$ do not usually exist. This section is concerned with the embedding of small subcategories of $\mathscr{A}/\mathscr{B}$ into a category which does have infinite sums, injective envelopes, and other desirable homological properties that $\mathscr{A}/\mathscr{B}$ lacks.

From this point on, it is assumed that $\mathscr{C}$ is a small Abelian subcategory of $\mathscr{A}/\mathscr{B}$ with the following properties:

(a) $\mathrm{Hom}_{\mathscr{C}}(A, B) = \mathrm{Hom}_{\mathscr{A}/\mathscr{B}}(A, B)$ for all $A, B \in \mathscr{C}$.

(b) $\mathscr{C}$ has enough projectives and enough injectives.

Such a category $\mathscr{C}$ may be obtained by taking as objects one copy of each group of cardinal not greater than some fixed infinite cardinal. The principal concern will be with an embedding of $\mathscr{C}$ into the category $\mathscr{L}(\mathscr{C}, \mathscr{A})$ of all left exact functors from $\mathscr{C}$ to $\mathscr{A}$. Recall that a map (or transformation) $u : F \to G$ of functors in $\mathscr{L}(\mathscr{C}, \mathscr{A})$ is a homomorphism $u(X) : F(X) \to$ $\to G(X)$ for each $X \in \mathscr{C}$ such that for $f : X \to Y$, the diagram

$$\begin{array}{ccc} F(X) & \xrightarrow{u(X)} & G(X) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ F(Y) & \xrightarrow{u(Y)} & G(Y) \end{array}$$

commutes. The set of such transformations, denoted here Trans (F, G), is a group in a natural way, and, in fact, with this definition of maps, $\mathscr{L}(\mathscr{C}, \mathscr{A})$ is an Abelian category with arbitrary infinite sums and products, and injective envelopes. (e.g. [8].) For $C \in \mathscr{C}$, the functor $\dot{C} : X \to \mathrm{Hom}(C, X)$ is left exact. A functor isomorphic to $\dot{C}$ is a *representable* functor. The functor $C \to \dot{C}$ is an exact functor from the category $\mathscr{C}^*$ dual to $\mathscr{C}$, to the category $\mathscr{L}(\mathscr{C}, \mathscr{A})$. In fact if

$$0 \to A \to B \to C \to 0$$

is exact in $\mathcal{C}^*$, then for $F \in \mathcal{L}(\mathcal{C}, \mathcal{A})$, the diagram

$$\begin{array}{ccccccc} 0 \to & \mathrm{Trans}(\dot{C}, F) & \to & \mathrm{Trans}(\dot{B}, F) & \to & \mathrm{Trans}(\dot{A}, F) \\ & \downarrow & & \downarrow & & \downarrow \\ 0 \to & F(C) & \to & F(B) & \to & F(A) \end{array}$$

is commutative with the vertical maps natural isomorphisms and the bottom row exact, F being left exact on $\mathcal{C}$. Thus the top row is exact. Further,

$$0 \to \mathrm{Trans}(F, \dot{A}) \to \mathrm{Trans}(F, \dot{B})$$

is also exact. If $u : F \to \dot{A}$, then since for each $X \in \mathcal{C}$, $u(X) : F(X) \to$ $\to \dot{A}(X) = \mathrm{Hom}_{\mathcal{C}}(A, X)$, u maps onto the transformation $v \in \mathrm{Trans}(F, \dot{B})$ for which $v(X)$ is the composition

$$F(X) \xrightarrow{u(X)} \dot{A}(X) = \mathrm{Hom}_{\mathcal{C}}(A, X) \to \mathrm{Hom}_{\mathcal{C}}(B, X) = \dot{B}(X).$$

Thus $u(X) = 0$ for all X if and only if this composition is 0. It follows that

$$0 \to \dot{A} \to \dot{B} \to \dot{C} \to 0$$

is exact in $\mathcal{L}(\mathcal{C}, \mathcal{A})$. Now $\mathrm{Trans}(\dot{A}, \dot{B}) = \dot{B}(A) = \mathrm{Hom}_{\mathcal{C}}(B, A) =$ $= \mathrm{Hom}_{\mathcal{C}^*}(A, B)$, so that $A \to \dot{A}$ is an exact embedding of $\mathcal{C}^*$ into $\mathcal{L}(\mathcal{C}, \mathcal{A})$, with the maps between objects in $\mathcal{C}^*$ 'the same as' maps between their images in $\mathcal{L}(\mathcal{C}, \mathcal{A})$. Also, a projective in $\mathcal{C}^*$ (i.e., an injective in $\mathcal{C}$) is a projective in $\mathcal{L}(\mathcal{C}, \mathcal{A})$. To see this, let

$$0 \to F \to G \xrightarrow{u} H \to 0$$

be exact in $\mathcal{L}(\mathcal{C}, \mathcal{A})$, and I injective in $\mathcal{C}$. Then

$$\begin{array}{ccccccc} 0 \to & \mathrm{Trans}(\dot{I}, F) & \to & \mathrm{Trans}(\dot{I}, G) & \to & \mathrm{Trans}(\dot{I}, H) \\ & \downarrow & & \downarrow & & \downarrow \\ 0 \to & F(I) & \to & G(I) & \to & H(I) \end{array}$$

is commutative with exact rows and the vertical maps natural isomorphisms. It suffices to show that

$$G(I) \xrightarrow{u(I)} H(I) \to 0$$

is exact. For any injective $I \in \mathcal{C}$, let $U(I) = \mathrm{Coker}\ u(I)$. Then U is an additive functor from the category $\mathcal{I}$ of injectives of $\mathcal{C}$ to $\mathcal{A}$. For $C \in \mathcal{C}$, let $0 \to C \to I_0 \xrightarrow{f} I_1$ be exact with I_0, I_1 injective. Defining $U(C) =$ $= \mathrm{Ker}\ U(f)$ yields an extension of U to a left exact functor in $\mathcal{L}(\mathcal{C}, \mathcal{A})$. The commutative diagram

$$\begin{array}{ccccccc} 0 \to & H(C) & \to & H(I_0) & \to & H(I_1) \\ & & & \downarrow & & \downarrow \\ 0 \to & U(C) & \to & U(I_0) & \to & U(I_1) \end{array}$$

yields a transformation of functors $v : H \to U$. Now the commutative diagram

$$\begin{array}{ccccc} 0 \to & G(C) & \to & G(I_0) \\ & \downarrow & & \downarrow \\ 0 \to & H(C) & \to & H(I_0) \\ & \downarrow & & \downarrow \\ 0 \to & U(C) & \to & U(I_0) \end{array}$$

has exact rows and last column exact. It follows that the composition $G(C) \to H(C) \to U(C)$ is 0. But $G \xrightarrow{u} H$ is an epimorphism, and $v \circ u = 0$. Thus $v = 0$, and so Coker $u(I) = 0$. Thus $u(I)$ is an epimorphism as desired.

One can show without difficulty that a map $F \to G$ in $\mathscr{L}(\mathscr{C}, \mathscr{A})$ is a monomorphism if and only if for each $A \in \mathscr{C}$, $F(A) \to G(A)$ is a monomorphism, and is an epimorphism if and only if for every injective $I \in \mathscr{C}$, $F(I) \to G(I)$ is an epimorphism.

The category $\mathscr{C}^*$ may be identified with its image in $\mathscr{L}(\mathscr{C}, \mathscr{A})$. Even though a projective in $\mathscr{C}^*$ is projective in the larger category $\mathscr{L}(\mathscr{C}, \mathscr{A})$, an injective in $\mathscr{C}^*$ may not be injective in $\mathscr{L}(\mathscr{C}, \mathscr{A})$.

THEOREM 5.1. *In $\mathscr{L}(\mathscr{C}, \mathscr{A})$, extensions of representable functors are representable.*

PROOF. Let

$$0 \to \dot{A} \to F \to \dot{B} \to 0$$

be exact in $\mathscr{L}(\mathscr{C}, \mathscr{A})$, with $A, B \in \mathscr{C}$. In $\mathscr{C}$, let

$$0 \to B \to I \to I/B \to 0$$

be exact with I injective. Then in $\mathscr{L}(\mathscr{C}, \mathscr{A})$,

$$0 \to (I\dot{/}B) \to \dot{I} \to \dot{B} \to 0$$

is exact with $\dot{I}$ projective. Hence

$$\begin{array}{ccccccccc} 0 \to & \mathrm{Trans}(\dot{B}, \dot{A}) & \to & \mathrm{Trans}(\dot{I}, \dot{A}) & \to & \mathrm{Trans}\big((I\dot{/}B), \dot{A}\big) & \to & \mathrm{Ext}_{\mathscr{L}(\mathscr{C}, \mathscr{A})}(\dot{B}, \dot{A}) \to 0 \\ & \downarrow & & \downarrow & & \downarrow & & \\ 0 \to & \mathrm{Hom}_{\mathscr{C}}(A, B) & \to & \mathrm{Hom}_{\mathscr{C}}(A, I) & \to & \mathrm{Hom}_{\mathscr{C}}(A, I/B) & \to & \mathrm{Ext}_{\mathscr{C}}(A, B) \to 0 \end{array}$$

is commutative with exact rows and vertical maps natural isomorphisms. Thus $\mathrm{Ext}_{\mathscr{L}(\mathscr{C}, \mathscr{A})}(\dot{B}, \dot{A}) \cong \mathrm{Ext}_{\mathscr{C}}(A, B) = \mathrm{Ext}_{\mathscr{C}^*}(B, A)$. Hence the exact sequence

$$0 \to \dot{A} \to F \to \dot{B} \to 0$$

is equivalent to an exact sequence

$$0 \to \dot{A} \to \dot{C} \to \dot{B} \to 0$$

and so $F \cong \dot{C}$. That is, F is representable.

THEOREM 5.2. *For* $F \in \mathscr{L}(\mathscr{C}, \mathscr{A})$ *and* $C \in \mathscr{C}$, $F(C)$ *is torsion-free divisible. That is,* $\mathscr{L}(\mathscr{C}, \mathscr{A})$ *is the category of left exact functors from* $\mathscr{C}$ *to the category of vector spaces over* Q.

PROOF. Suppose $F(C)$ is not torsion-free for some $C \in \mathscr{C}$. Then the composition of F with the left exact functor that takes a group onto its p-socle is, for some p, a non-zero functor $G \in \mathscr{L}(\mathscr{C}, \mathscr{A})$. The representable functors in $\mathscr{L}(\mathscr{C}, \mathscr{A})$ form a family of generators [8]. Thus there exists a representable functor X and a non-zero map $\dot{X} \to G$. For some $D \in \mathscr{C}$, $\dot{X}(D) \to G(D)$ is not zero. But $\dot{X}(D) = \mathrm{Hom}_{\mathscr{C}}(X, D) = Q \otimes \mathrm{Hom}(X, D)$ is torsion-free divisible, and $G(D)$ is bounded by p. This contradiction shows that $F(C)$ is torsion-free for all $C \in \mathscr{C}$. Now $F(C) \cong \mathrm{Trans}(\dot{C}, F)$ being torsion-free implies that the composition of F with the functor E that takes a group onto its torsion-free divisible part (that functor is not left exact) is left exact. For any $D \in \mathscr{C}$ and $u \in \mathrm{Trans}(\dot{C}, F)$, the image of $u(D) : \dot{C}(D) \to F(D)$ is torsion-free divisible, since $\dot{C}(D) = Q \otimes \mathrm{Hom}(C, D)$ and $F(D)$ is torsion-free. Hence $\mathrm{Im}\big(u(D)\big) \subset (E \circ F)(D)$, and u induces a transformation $u' : \dot{C} \to E \circ F$. Clearly $\mathrm{Trans}(\dot{C}, F) \to \mathrm{Trans}(\dot{C}, E \circ F) : u \to u'$ is a monomorphism. Thus $0 \to F(C) \to (E \circ F)(C)$ is exact. But $(E \circ F)(C)$ is the torsion-free divisible part of $F(C)$. Therefore $F(C)$ is torsion-free divisible as asserted.

THEOREM 5.3. *For* $F, G \in \mathscr{L}(\mathscr{C}, \mathscr{A})$, $\mathrm{Trans}\ (F, G)$ *is torsion-free divisible.*

PROOF. For any subobject F_a of F, there is a representable functor $\dot{X}_a$ and a map f_a such that the composition $\dot{X}_a \xrightarrow{f_a} F \to F/F_a$ is not zero, since the representable functors form a family of generators. Hence there is a map $\sum_a \dot{X}_a \xrightarrow{\sum_a f_a} F$ which is an epimorphism. The resulting exact sequence

$$0 \to K \to \sum_a \dot{X}_a \to F \to 0$$

yields

$$0 \to \mathrm{Trans}(F, G) \to \mathrm{Trans}\Big(\sum_a \dot{X}_a, G\Big) \to \mathrm{Trans}(K, G)$$

exact. $\mathrm{Trans}\big(\sum_a \dot{X}_a, G\big) \cong \prod_a \mathrm{Trans}(\dot{X}_a, G) \cong \prod_a G(X_a)$ is torsion-free divisible. Hence $\mathrm{Trans}(F, G)$ is torsion-free for all $F, G \in \mathscr{L}(\mathscr{C}, \mathscr{A})$. In particular, $\mathrm{Trans}(K, G)$ is torsion-free, and this makes $\mathrm{Trans}(F, G)$ torsion-free divisible as asserted.

LEMMA 5.4. *If* $F \in \mathscr{L}(\mathscr{C}, \mathscr{A})$ *is an epimorphic image of a representable functor and a subobject of a representable functor, then* F *is a representable functor.*

PROOF. Suppose

$$0 \to K \to \dot{X} \to F \to 0$$

and

$$0 \to F \to \dot{Y} \to G \to 0$$

are exact. Then

$$0 \to K \oplus \dot{Y} \to \dot{X} \oplus \dot{Y} \to F \to 0$$

is exact, so that F is an endomorphic image of $\dot{X} \oplus \dot{Y}$, which is representable. Since $\mathrm{Hom}_{\mathcal{C}}(X \oplus Y, X \oplus Y) \cong \mathrm{Trans}(\dot{X} \oplus \dot{Y}, \dot{X} \oplus \dot{Y})$, it follows that F is representable.

This lemma yields immediately

COROLLARY 5.5. *If $\dot{X} = \sum_a F_a$ or $\prod_a F_a$ in $\mathcal{L}(\mathcal{C}, \mathcal{A})$, then each F_a is representable.*

Thus an object in $\mathcal{C}$ has no new direct decompositions in $\mathcal{L}(\mathcal{C}, \mathcal{A})$. However, an object in $\mathcal{C}$ may gain new subobjects in $\mathcal{L}(\mathcal{C}, \mathcal{A})$. This follows from Gabriel [8, Chapter III, Proposition 4, and Chapter II, Proposition 10]. The significance of these new subobjects for the study of the 'quasi-structure' of a group is not too clear.

In $\mathcal{L}(\mathcal{C}, \mathcal{A})$, it is easy to see that $Z(p^\infty)$ and Z are simple objects. If Q_p denotes the subgroup of Q of elements whose denominators are powers of p, then

$$0 \to Z \to Q_p \to Z(p^\infty) \to 0$$

is exact in $\mathcal{C}$, so that

$$0 \to \dot{Z}(p^\infty) \to \dot{Q}_p \to \dot{Z} \to 0$$

is exact in $\mathcal{L}(\mathcal{C}, \mathcal{A})$. Further, $\dot{Z}(p^\infty) \subseteq\!\!\!\!/\!\cong \dot{Z}$, and $\mathrm{Ext}_{\mathcal{L}(\mathcal{C}, \mathcal{A})}(\dot{Z}, \dot{Z}(p^\infty) \neq 0$. Thus in a complete Abelian category, the group of extensions of two non-isomorphic simple objects may be non-zero.

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